A continuous extension of a load-share reliability model based on a condition of the residual lifetime conservation

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Abstract: The reliability behavior of systems under changeable load is studied in the paper. A case when the load changes continuously in time is considered, i.e., a continuous extension of the discrete load-share reliability models is proposed. The main assumption used in the paper is the so-called condition of the residual lifetime conservation of the system, which is equivalent to the condition of continuity of the reliability function. Special cases of the exponential and Weibull probability distributions of time to failure are provided in detail. Various numerical examples are provided to illustrate the reliability behavior of systems under the continuously changeable load.

Keywords: reliability; system load-share model; failure rate; survivor function; exponential distribution; mean time to failure.

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1 Introduction

A system during its functioning may work under different changeable conditions which impact on the system lifetime and the system reliability behavior. One of the possible reasons of the system working condition changes and of the system reliability behavior changes is the changeable load on the system or its components. One of the important application examples of the changeable load is electrical networks whose load strongly depends on outdoor temperature and directly leads to the reliability behavior changes. For example, in Lapland (Finland), where electrical heating is of great importance and deviations in outdoor temperature are large, the load increase caused by outdoor temperature may be up to 100% compared to the load at normal temperature (Meldorf et al. (2007)). Valor et al. (2001) indicate that, by seasons, the electricity load shows maximum values in winter and summer and minimum values in the transition periods, that is, spring and autumn. This trend is very close to the one shown by electricity consumption. In January, maximum consumption coincides with minimum temperatures. In March–April, there is a transition with a near-constant consumption level until May, while temperatures are increasing. Beyond this point, temperature increase causes an electricity load increase in the summer period, except in August. In September–October, another transition period is observed, with a constant demand level with decreasing temperatures. From November the demand increases together with the temperature decrease, and the cycle is repeated again. As pointed out by Tanrioven and Alam (2005), studies that quantify power system reliability are often limited to constant transmission rates, covering two weather conditions, namely normal and adverse weather. In reality, however, the probability of a system or a component failure varies from time to time dependent upon factors such as a change of environmental conditions, demand variation and random failures in the system. The correct reliability analysis requires to take into account the different conditions.

Another reason of the load changes is some stochastic dependency among the system's units, which means that the reliability behavior of some units in the system depends on a state (working or failed) of other units. This dependency and the change of the load conditions can be modelled by means of the so-called load-share models. According to load-share models, failure rates of units in a system depend on states of other units of the system. A crucial point of the models is the rule that governs how failure rates of units change after failures of other system units. One of the pioneering works devoted to load-share models applied in the textile industry was proposed by Daniels (1945) in 1945. Last decades, many authors contribute to the load-share models, for instance, Bebbington et al. (2007);

Coleman (1957); Ross (1984); Durham et al. (1995); Lynch (1999); Kim and Kvam (2004); Kvam and Pena (2005); Stefanescu and Turnbull (2006). It should be noted that the above works are a very small part of a huge number of papers devoted to load-share models due to their importance in many applications.

The discrete load-share reliability models with the piecewise constant load have been studied by Gurov and Utkin (2012). These models are discrete because the load changes occur step-wise at discrete time instances. In contrast to many available models, the proposed models exploit an important assumption which defines these models. This assumption is the so-called "condition of the residual lifetime conservation" of a system, which is equivalent to the condition of continuity of the cumulative distribution function of time to failure or the reliability (survivor) function.

The discrete models with the condition of the residual lifetime conservation cover only a part of various systems with the load changes. At the same time, many systems are functioning under continuously changed conditions, for instance, under changes of the temperature around the systems as it has been shown above. In this case, the load factor changes continuously. One of the simplest ways for analyzing the system is to discretize the continuous function of the load change and to apply the discrete reliability models proposed by Gurov and Utkin (2012). However, this way might lead to extremely complex expressions when times to failure are governed by non-exponential probability distributions. If changes are described in a closed form as continuous functions, the attempt to discretize the continuous function often hides some interesting system reliability behavior peculiarities which could be explicitly observed or revealed when we study continuous models. Moreover, the differential equations are derived in the case of continuous models, which can be numerically solved by means of various methods. Therefore, our aim is to extend the discrete models on the case of continuous changes.

Let $P(t)$ be the reliability of a system under normal (or initial) conditions of its usage. The main problem of the system analysis is to determine the reliability of the system by taking into account the possible continuous changes of the load. Our initial information about the changeable load will be represented by means of the load rate as a measure of the load. Changes of the system working conditions may reduce or increase the working time of the system. Moreover, by changing the load, we can assert that the system failure rate in this case also changes. This implies that the system reliability can not be measured by $P(t)$. Therefore, we have to define another measure $P_c(t)$ which takes into account the working condition changes.

A general method for defining and computing the reliability function $P_c(t)$ is proposed in the paper. Moreover, we investigate in the paper how types of probability distributions of time to failure impact on the reliability measures of the system with different types of continuous load rate functions. The measures for analyzing the system reliability are the reliability function $P_c(t)$ and the mean time to failure T_c.

The paper is organized as follows. In Section 2, we consider a discrete load-share model and introduce the condition of the residual lifetime conservation of the system. The results provided in Section 2 are extended on a case of continuous changes of load in Section 3. It is shown in this section that the problem of computing $P_c(t)$ is reduce to computing the so-called shift function by solving a differential equation. A special case when the system time to failure is governed by the exponential probability distribution is studied in Section 4. The same study by Weibull distribution of time to failure is given in Section 5. A number of numerical examples illustrating different load conditions of the system working can be found in Section 6.

2 Discrete load-share models

In order to give an example of discrete load-share models, we consider a parallel system consisting of n units. It is supposed that $n-1$ units are redundant. After failure of a unit, other units are under the increased load which is measured by the so-called load factor or the load rate denoted k that is $k = n/(n-1)$ for the considered parallel system. After failure of the second unit, the load increases again and the load factor becomes to be $k = n/(n-2)$. By continuing the consideration of failures, we can say that the load on the last single working unit increases n times after failures of $n-1$ units and the load factor is now $k = n$. Generally, the load rate may be arbitrary at different time moments.

The time moments $t_1, t_2, ..., t_{n-1}$ of the load changes may be deterministic or random. Therefore, the system behavior after the load changes a priori may be unknown because it depends on the time moments of changes. From the mathematical point of view, the above means that the system lifetime distribution changes under the load.

Below, we study the case when the time moments of the load changes are deterministic because it is assumed that the load changes are caused by ambient, environmental, external conditions, for instance, outdoor temperature. However, the proposed approach can be easily extended on the case of random load changes by using the rule of total probability.

Let us consider first a case when the load on a system changes only once at time t_1 such that the load rate changes from the initial (normal) value 1 before time t_1 to some new value k corresponding the load after time t_1. Let $P_k(t)$ be the reliability of the system under condition that the load has changed and became to be k. The main idea underlying the computation of the reliability measure $P_c(t)$ after the load changes is the so-called "condition of the residual lifetime conservation" of the system. Namely, it can be written as

$$P_c(t) = \begin{cases} P(t), & \text{if } t < t_1, \\ P_k(t-x), & \text{if } t \geq t_1, \end{cases} \tag{1}$$

where the value of the "shift" x is chosen in such a way that $P_c(t)$ is continuous (without jumps).

It is equivalent to the condition of continuity of the cumulative distribution function of time to failure or the reliability (survivor) function.

Rather simple expressions for $P_k(t)$ can be obtained if we link the load rate k and the system failure rate. In particular, suppose that the load increases in k times as the system failure rate increases in k times. Then there holds

$$P_k(t) = P^k(t). \tag{2}$$

A case when the load changes occur step-wise at discrete time instances $t_1, t_2, ..., t_n$ and the loads between these instances are constant has been studied by Gurov and Utkin (2012). The load rates at these instances are $k_1, k_2, ..., k_n$, respectively. We assume $t_0 = 0,\ t_{n+1} = +\infty,\ k_0 = 1$.

Then the reliability of the system, taking into account the changeable load, is

$$P_c(t) = P_{k_i}(t - x_i), \tag{3}$$

if $t_i \leq t < t_{i+1},\ i = 0, 1, 2, ..., n$.

The function $P_{k_i}(t)$ is the reliability function of the system under condition that the corresponding load rate is k_i. Parameters x_i, $i = 1, 2, ..., n$, are computed by means of the recurrent algorithm from the following equation:

$$P_{k_i}(t_i - x_i) = P_{k_{i-1}}(t_i - x_{i-1}). \tag{4}$$

At that $x_0 = 0$.

3 The transition from the discrete load-share model to a continuous model

Let $k(t)$ be a differentiable function of time characterizing the load rate. The condition $k(0) = 1$ has to be valid at time $t = 0$. This is an analogue of the normalizing condition which means that the system starts working under a "normal" load condition. Suppose that the system has the reliability function $P(t) = e^{-\Lambda(t)}$ and the corresponding cumulative failure rate function $\Lambda(t) = \int_0^t \lambda(\tau)\, d\tau$.

We assume below that the change of the load in k times leads to change of the system failure rate in k times. Then there holds $P_k(t) = P^k(t)$ (see (2)). Moreover, it follows from (4) that there holds

$$e^{-k_i \Lambda(t_i - x_i)} = e^{-k_{i-1} \Lambda(t_i - x_{i-1})}.$$

Hence, we can write

$$k_i \Lambda(t_i - x_i) = k_{i-1} \Lambda(t_i - x_{i-1}).$$

Here k_i is the load rate at time t_i, i.e., $k_i = k(t_i)$.

If we denote $\Delta t_i = t_i - t_{i-1}$ the time between the time moments t_i and t_{i-1}, then the load rate at time t_{i-1} can be written as $k_{i-1} = k(t_{i-1}) = k(t_i - \Delta t_i)$. Then we can write for arbitrary Δt and t the following equalities:

$$k(t + \Delta t)\, \Lambda(t + \Delta t - x(t + \Delta t)) = k(t)\, \Lambda(t + \Delta t - x(t))$$

and

$$\begin{aligned}&(k(t + \Delta t) - k(t))\, \Lambda(t + \Delta t - x(t + \Delta t)) \\ &= k(t)\, (\Lambda(t + \Delta t - x(t)) - \Lambda(t + \Delta t - x(t + \Delta t))).\end{aligned}$$

If we divide both sides of the last equality into Δt and use transition to the limit by $\Delta t \to 0$, then we get

$$k'(t)\,\Lambda(t - x(t)) = k(t)\,\lambda(t - x(t))\,x'(t)\,.$$

Here $k'(t) = \mathrm{d}k(t)/\mathrm{d}t$ and $x'(t) = \mathrm{d}x(t)/\mathrm{d}t$. Hence

$$x'(t) = \frac{k'(t)}{k(t)} \frac{\Lambda(t - x(t))}{\lambda(t - x(t))}. \tag{5}$$

It follows from (3) that the reliability function of the system, taking into account the continuously varying load, is

$$P_c(t) = P_{k(t)}(t - x(t))\,. \tag{6}$$

The initial condition here is $x(0) = 0$. The function $k(t) \equiv 1$ corresponds to the lack of load. Here we mean the lack of the additional load different from the initial one. In this case, we can write $x(t) \equiv 0$.

If the system is under the continuously varying load with the load rate $k(t)$, then the system reliability function is

$$P_c(t) = e^{-M(t)} = e^{-k(t)\Lambda(t - x(t))},$$

where the "shift" function $x(t)$ satisfies the differential equation (5); $M(t)$ is the cumulative failure rate under the changed load.

This implies that the cumulative failure rate $M(t)$ is of the form:

$$M(t) = k(t)\,\Lambda(t - x(t))\,. \tag{7}$$

It follows from (5) that the shift function $x(t)$ depends on the system lifetime probability distribution under the normal working state (the cumulative failure rate $\Lambda(t)$ and the failure rate $\lambda(t)$). It also depends on the system load and the load rate function.

This equation can be solved in the explicit form only for some special cases. Generally, a corresponding numerical method is required for computing its solution.

Let us point out the following properties of the function $x(t)$ and other functions associated with it.

1. If the system load rate function is constant in the time interval $[a, b]$, then the shift function $x(t)$ is also constant and its value is $x(a)$. This property directly follows from the equation (3).

2. The system failure rate taking into account the load is of the form (we assumed that change of the load in k times leads to change of the system failure rate in k times):

$$\mu(t) = \frac{\mathrm{d}M(t)}{\mathrm{d}t} = k(t)\,\lambda(t - x(t))\,. \tag{8}$$

The failure rate corresponds to the function $P_c(t)$. Equation (8) looks like the cumulative failure rate $M(t)$ in (7). Indeed, by differentiating the function $M(t)$, we get

$$\mu(t) = k'(t)\,\Lambda(t - x(t)) + k(t)\,\lambda(t - x(t))\,(1 - x'(t))\,.$$

Hence

$$\mu(t) = k'(t)\,\Lambda(t - x(t)) + k(t)\,\lambda(t - x(t)) - k(t)\,\lambda(t - x(t))\,x'(t)\,.$$

By applying the equation (5), we obtain (8).

The differential equation (5) can not be generally solved in the explicit form. However, we are able to explicitly solve it in some important special cases

4 The exponential distribution

Assume that the time to failure under normal conditions of functioning is governed by the exponential distribution with the reliability function $P(t) = e^{-\lambda t}$. This implies that the system under the continuous load $k(t)$ has the reliability function $P_c(t) = e^{-\lambda K(t)}$, where the function $K(t) = \int_0^t k(\tau)\,d\tau$ reflects the system cumulative load rate. In other words, the following equality can be written for the case of the exponential distribution of time to failure

$$P_c(t) = P(K(t))\,. \tag{9}$$

Let us prove this proposition. In accordance with (7), we can write $M(t) = k(t)\,\Lambda(t - x(t))$ and for the exponential case distribution $\Lambda(t) = \lambda t$. Then it follows from the differential equation (5)

$$x'(t) = \frac{k'(t)}{k(t)}\,(t - x(t))\,.$$

Hence $(k(t)\,x(t))' = tk'(t)$. By integrating the above equation under condition $x(0) = 0$, we get

$$k(t)\,x(t) = \int_0^t tk'(t)\,dt,$$

or $k(t)\,x(t) = tk(t) - K(t)$. Hence, we can write $k(t)\,(t - x(t)) = K(t)$. The above implies $P_c(t) = e^{-M(t)} = e^{-\lambda K(t)}$, as was to be proved.

5 The Weibull distribution

Assume that $\Lambda(t)$ is the power function of the form $\Lambda(t) = \lambda t^n$. In this case, the differential equation (5) is

$$x'(t) = \frac{k'(t)}{k(t)} \cdot \frac{t - x(t)}{n}$$

with the initial condition $x(0) = 0$. It is easy to show that a solution of this Cauchy problem is the function

$$x(t) = t - k^{-\frac{1}{n}}(t) \int_0^t k^{\frac{1}{n}}(\tau)\, d\tau. \tag{10}$$

Therefore, we can write

$$M(t) = \lambda k(t)(t - x(t))^n = \lambda \left(\int_0^t k^{\frac{1}{n}}(\tau)\, d\tau \right)^n = \Lambda \left(\int_0^t k^{\frac{1}{n}}(\tau)\, d\tau \right). \tag{11}$$

Hence

$$P_c(t) = P \left(\int_0^t k^{\frac{1}{n}}(\tau)\, d\tau \right). \tag{12}$$

Simultaneously, we have now the explicit expression for the system reliability function under the continuous load and by the power function $\Lambda(t)$. Namely, there holds

$$P_c(t) = e^{-M(t)},$$

where the cumulative failure rate taking into account the load is determined from (11).

Let us consider this problem in detail. The cumulative failure rate for the Weibull distribution with parameters α and β is $\Lambda(t) = (t/\beta)^\alpha$. Therefore, we can write $n = \alpha$ and $\lambda = 1/\beta^\alpha$. According to (10), we get the shift function

$$x(t) = t - k^{-\frac{1}{\alpha}}(t) \int_0^t k^{\frac{1}{\alpha}}(\tau)\, d\tau. \tag{13}$$

Moreover, according to (9), we get the cumulative failure rate

$$M(t) = \left(\frac{1}{\beta} \int_0^t k^{\frac{1}{\alpha}}(\tau)\, d\tau \right)^\alpha. \tag{14}$$

Suppose that the load rate is a power function of the form $k(t) = (a + bt)^m$. Here the value of m may be positive as well as negative. Then the integral in (13) or in (14) is simply computed. As a result, we obtain

$$\int_0^t k^{\frac{1}{\alpha}}(\tau)\, d\tau = \int_0^t (a + bt)^{\frac{m}{\alpha}}(\tau)\, d\tau = \begin{cases} \frac{(a+bt)^{1+\frac{m}{\alpha}} - a^{1+\frac{m}{\alpha}}}{b\left(1+\frac{m}{\alpha}\right)}, & m \neq -\alpha, \\ \frac{\ln(a+bt) - \ln a}{b}, & m = -\alpha. \end{cases}$$

Hence

$$x(t) = \begin{cases} t - \frac{a+bt-a^{1+\frac{m}{\alpha}}(a+bt)^{-\frac{m}{\alpha}}}{b\left(1+\frac{m}{\alpha}\right)}, & m \neq -\alpha, \\ t - (a+bt)^{-\frac{m}{\alpha}} \frac{\ln(a+bt)-\ln a}{b}, & m = -\alpha, \end{cases} \tag{15}$$

and

$$M(t) = \begin{cases} \left(\frac{(a+bt)^{1+\frac{m}{\alpha}} - a^{1+\frac{m}{\alpha}}}{b\beta\left(1+\frac{m}{\alpha}\right)} \right)^{\alpha}, & m \neq -\alpha, \\ \left(\frac{\ln(a+bt) - \ln a}{b\beta} \right)^{\alpha}, & m = -\alpha. \end{cases} \quad (16)$$

In sum, a simple expression for the reliability function $P_c(t) = e^{-M(t)}$ of a system with the power load can be obtained when its time to failure has the Weibull distribution with parameters α and β. If the cumulative failure rate $\Lambda(t)$ is arbitrary, then the equation (5) can be solved by numerical methods in order to compute the shift function $x(t)$ and the system reliability $P_c(t)$.

6 The model analysis under various conditions

Below we consider various numerical examples illustrating the proposed continuous models, which allow us to analyze some peculiarities of the proposed models. A software program UNCEASING_LOAD has been developed for arbitrary smooth (continuously differentiable) functions $k(t)$. It allows us to calculate the function $x(t)$ as well as the system reliability $P_c(t)$ taking into account the load changes for cases when an explicit solution expression to equation (15) can not be obtained.

Differential equations are solved by means of the Runge-Kutta method with the adaptive step size. In parallel, the system mean time to failure T_c is computed. Numerical results obtained by means of the software program UNCEASING_LOAD and equation (15) agree within four digits for two cases considered below (Examples 6.1 and 6.2).

6.1 The function $x(t)$ and the Weibull distribution

Explicit expressions (15) for computing $x(t)$ are obtained for some special cases under condition that the system time to failure has the Weibull distribution with parameters α and β. Below we provide the results of computing the function $x(t)$ by using the proposed software and by using (13). It is assumed that the system time to failure has the Weibull distribution with the expectation $m = 10$ h and the mean square deviation $\sigma = 2$ h. The parameters α and β in this case are determined as

$$m = \beta \cdot \Gamma\left(1 + \frac{1}{\alpha}\right), \ \sigma = \beta \cdot \sqrt{\Gamma\left(1 + \frac{2}{\alpha}\right) - \Gamma^2\left(1 + \frac{1}{\alpha}\right)},$$

where $\Gamma(t) = \int_0^{+\infty} x^{t-1} e^{-x} dx$ is the standard Gamma function.

Hence $\alpha = 5.7966$, $\beta = 10.7998$. Below we consider three different load rates.

Example 6.1: $k(t)$ is increasing: $k(t) = 1 + 0.1t$. The obtained shift function $x(t)$ is depicted in Fig. 1. One can see from Fig. 1 that $x(t)$ is increasing and takes only non-negative values. This implies that the system reliability function taking into account the increased load conditions is under the initial reliability function and is dropping quickly.

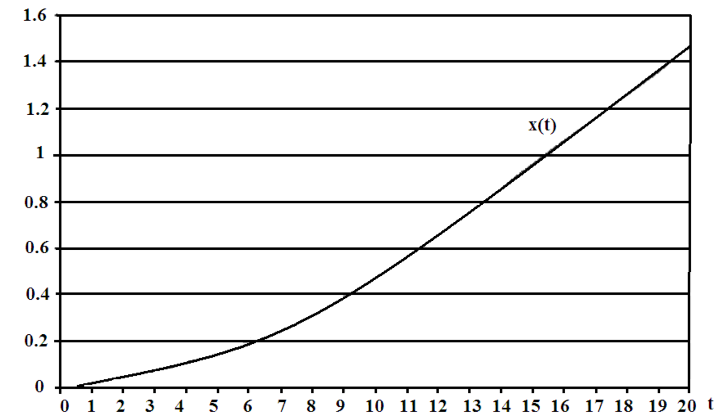

Figure 1 The increasing shift function

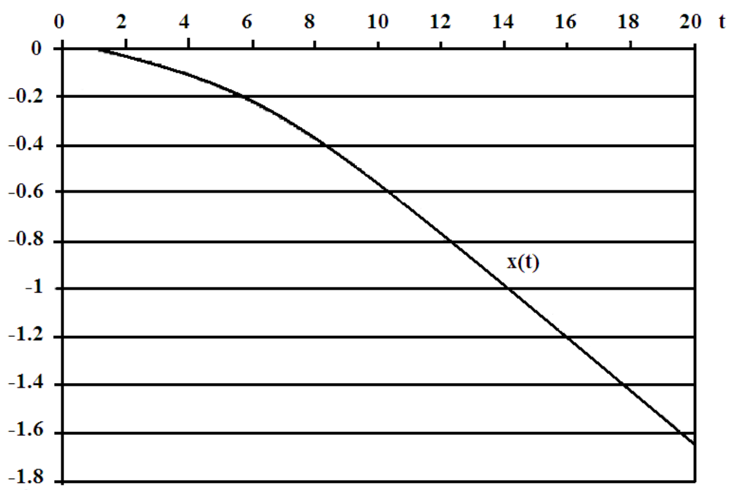

Figure 2 The decreasing shift function

Example 6.2: $k(t)$ is decreasing: $k(t) = (1 + 0.1t)^{-1}$. The function $x(t)$ is depicted in Fig. 2. It is decreasing and takes non-positive values. This implies that the system reliability function taking into account the decreased load conditions is up the initial reliability function.

Example 6.3: $k(t)$ is non-monotone: $k(t) = 1 + 0.5 \sin t$. The function $x(t)$ is depicted in Fig. 3. It has an oscillatory form. Moreover, it takes positive as well as negative values. The relationship of the initial reliability function and the function taking into account the changeable load conditions will be shown below (see Fig. 9).

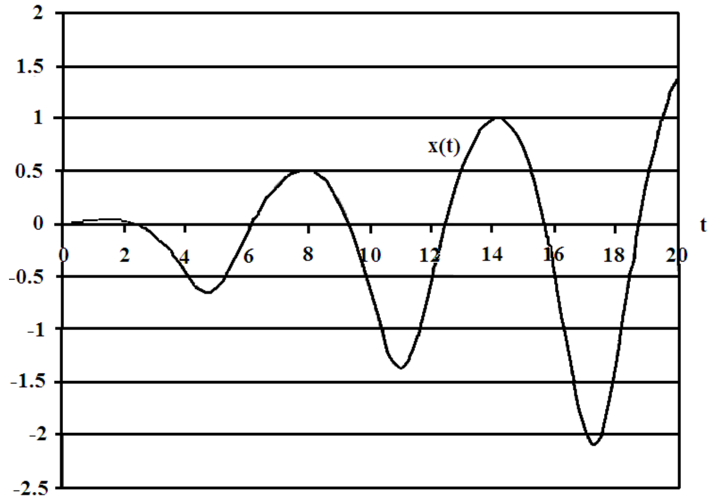

Figure 3 The shift function of the oscillatory form

Now we consider examples illustrating how the system reliability functions and the mean time to failure depend on a type of probability distributions of time to failure and its parameters.

6.2 The system reliability under the increasing load

In all examples of the subsection, we consider the linear load function $k(t) = 1 + 0.1t$. In this case, the system cumulative load rate is $K(t) = t + 0.1t^2/2$.

Example 6.4: It is assumed that the system time to failure is governed by the exponential distribution with the parameter $\lambda = 0.1$ h^{-1}. The system reliability function is

$$P_c(t) = e^{-\lambda K(t)} = e^{-0.1\left(t+0.05t^2\right)}.$$

Functions $P(t)$ and $P_c(t)$ are depicted in Fig. 4.

Example 6.5: It is assumed that the system time to failure is governed by the Weibull distribution with the expectation $m = 10$ h and the mean square deviation $\sigma = 2$ h. Functions $P(t)$ and $P_c(t)$ are depicted in Fig. 5.

One can see that the reliability functions corresponding to the exponential and Weibull distributions are quite different in spite of the identical load conditions and identical expectations. Nevertheless, the mean square deviation of the exponentially distributed time to failure is $1/\lambda = 10$ h. This fact explains such a reliability function relationship.

Example 6.6: It is assumed that the system time to failure is governed by the gamma distribution with the expectation $m = 10$ h and mean square deviation $\sigma = 2$ h. Functions $P(t)$ and $P_c(t)$ are depicted in Fig. 6.

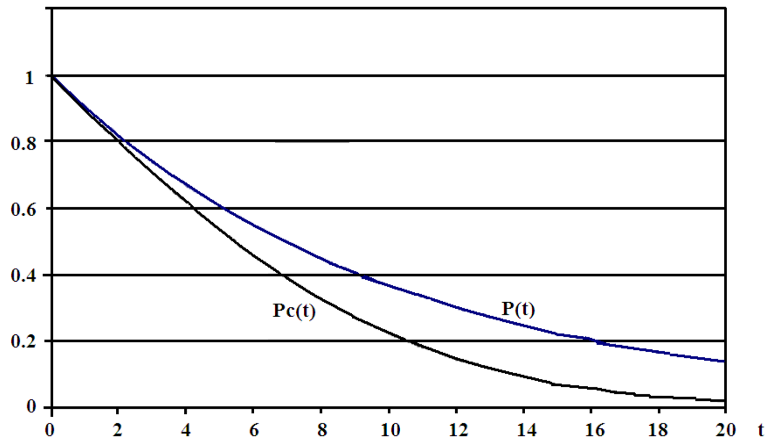

Figure 4 The reliability functions by the exponential distribution of time to failure and the linear load rate

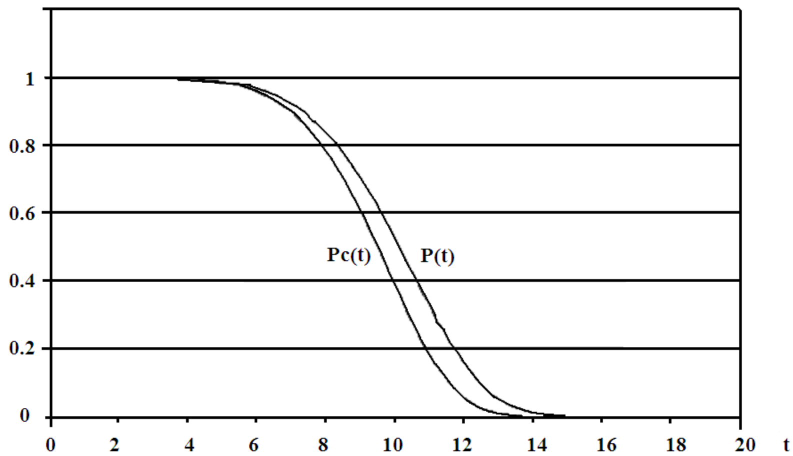

Figure 5 The reliability functions by the Webull distribution of time to failure and the linear load rate

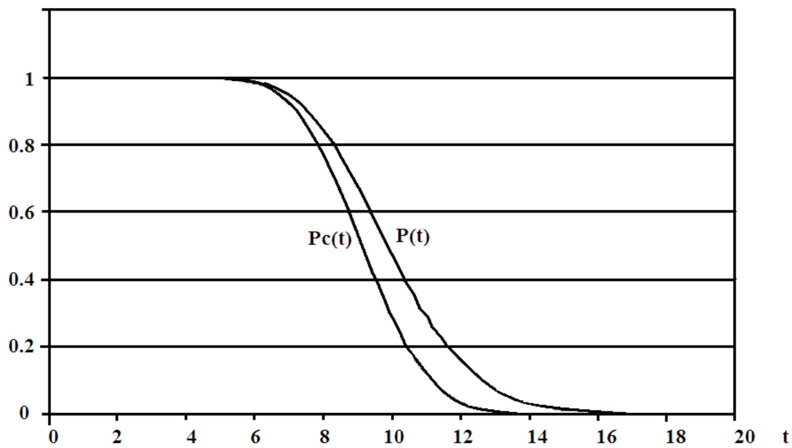

Figure 6 The reliability functions by the gamma distribution of time to failure and the linear load rate

Similar functions are obtained by assuming that the time to failure has the normal distribution under the same conditions. One can see that there is only a small difference between the three last cases when two-parametric probability distributions (Weibull, gamma, normal) with the same parameters are considered. Of course, the type of probability distributions of time to failure impacts on the reliability behavior, but not so strongly when the parameters of the distributions are identical.

Example 6.7: It is assumed that the system time to failure is governed by the Rayleigh distribution with the expectation $m = 10$ h. Functions $P(t)$ and $P_c(t)$ are depicted in Fig. 7. The mean square deviation σ of the Rayleigh distributed time to failure is $m\sqrt{2/\pi} = 7.98$ h. We have an intermediate case between the exponential distribution with $\sigma = 10$ h and the two-parametric distributions with $\sigma = 2$ h. This is one of the reasons explaining the reliability function behavior by the Rayleigh distributed time to failure. The mean time to failure without taking into account the load factor is $m = 10$ h. The mean time to failure as shown in Table 1 decreases under the increased load. We can see again that the exponential distribution can be regarded as a “worse” distribution in the sense that it leads to the smallest mean time to failure. This feature of the exponential distribution has been proved for the discrete case of the changeable load (see Gurov and Utkin (2012)).

6.3 The system reliability under the decreasing load

Example 6.8: It is assumed that the system time to failure is governed by the Weibull distribution with the expectation $m = 10$ h and the mean square deviation $\sigma = 2$ h. The load function is of the form $k(t) = (1 + 0.1t)^5$. Functions $P(t)$ and $P_c(t)$ are depicted in Fig. 8. It can be seen that the load change impacts very

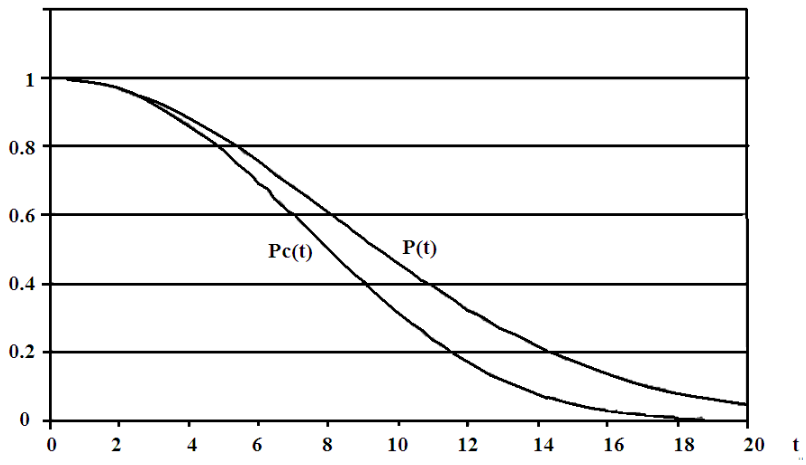

Figure 7 The reliability functions by the Rayleigh distribution of time to failure and the linear load rate

Table 1 The mean time to failure by different probability distributions

Distribution	T_c, h
Exponential	6.50
Gamma	9.15
Normal	9.34
Rayleigh	8.22
Weibull	9.37

strongly on the system reliability and $P_c(t) \geq P(t)$ for all t. The same conclusion is valid for the mean time to failure, namely, $T = 10$ h, $T_c = 15.43$ h.

6.4 The system reliability under the non-monotone load

Example 6.9: It is assumed that the system time to failure is governed by the Rayleigh distribution with the expectation $m = 10$ h. The function $k(t)$ is non-monotone: $k(t) = 1 + 0.8 \sin t$. It corresponds to the seasonal load changes caused by outdoor temperature. Functions $P(t)$ and $P_c(t)$ are depicted in Fig. 9. The function $x(t)$ is shown in Fig. 10. The values of the mean time to failure are $T = 10$ h, $T_c = 9.66$ h. The load rate can be regarded as an alternating process. One can see from Fig. 9 that the curves of the functions $P_c(t)$ and $P(t)$ intersect each other at several points corresponding to $k(t)$.

Example 6.10: It is assumed that the system time to failure is governed by the normal distribution with the expectation $m = 10$ h and the mean square deviation $\sigma = 2$ h. The load rate has the oscillatory behavior $k(t) = 7 + 6 \sin\left(t - \frac{\pi}{2}\right)$ such that $k(t) \geq 1$. The reliability functions $P(t)$ and $P_c(t)$ are shown in Fig. 11. The function $x(t)$ is shown in Fig. 12. The values of the mean time to failure are $T = 10$

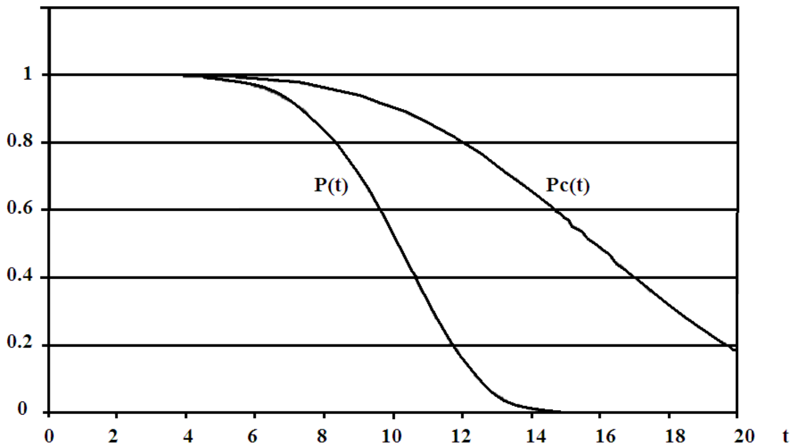

Figure 8 The reliability functions by the Weibull distribution of time to failure and the non-linear load rate

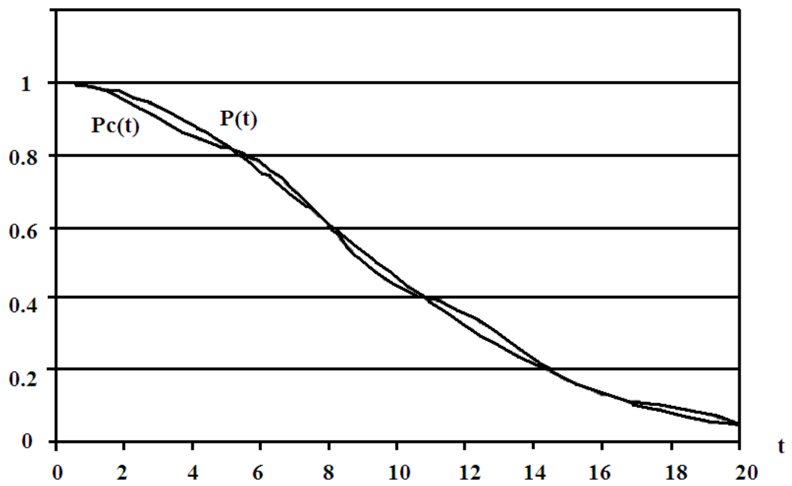

Figure 9 The reliability functions by the Rayleigh distribution of time to failure and the sinusoidal load rate

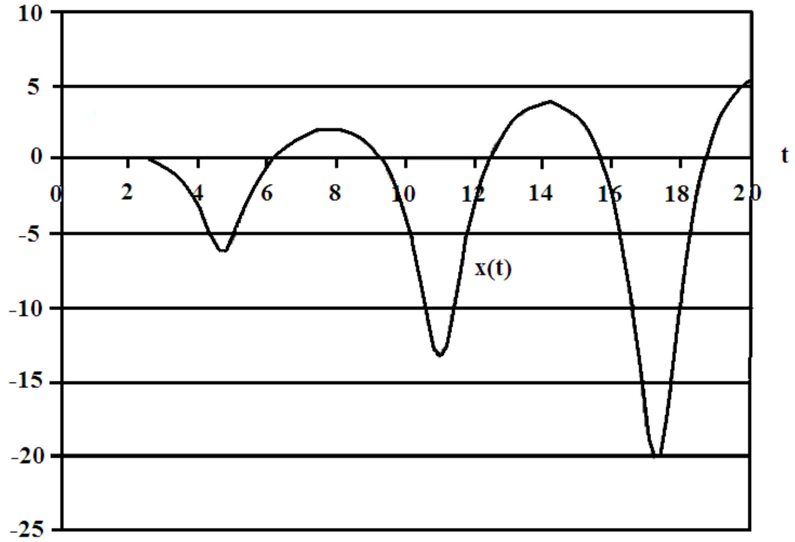

Figure 10 The shift function by the sinusoidal load rate and the by the Rayleigh distribution of time to failure

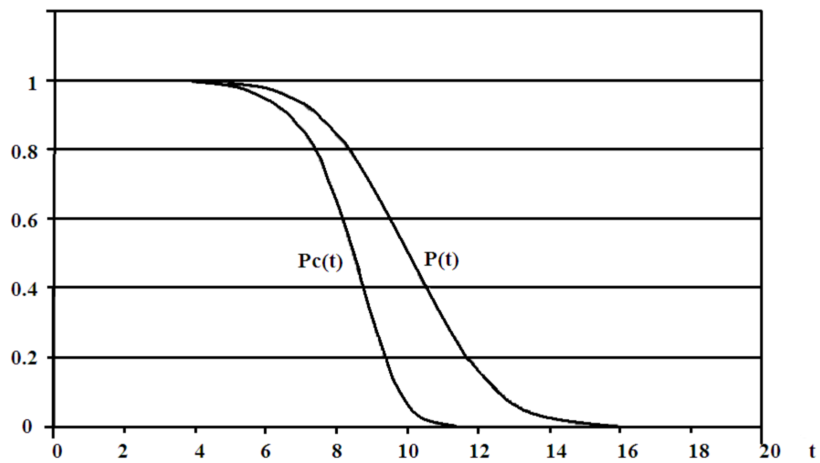

Figure 11 The reliability functions by the normal distribution of time to failure and the sinusoidal load rate

h and $T_c = 9.66$ h. Here the curve $P_c(t)$ is below the curve $P(t)$ because the load rate is larger than 1.

7 Conclusion

The continuous extension of the load-share reliability models has been studied in the paper. The main feature of the considered models is applying the condition

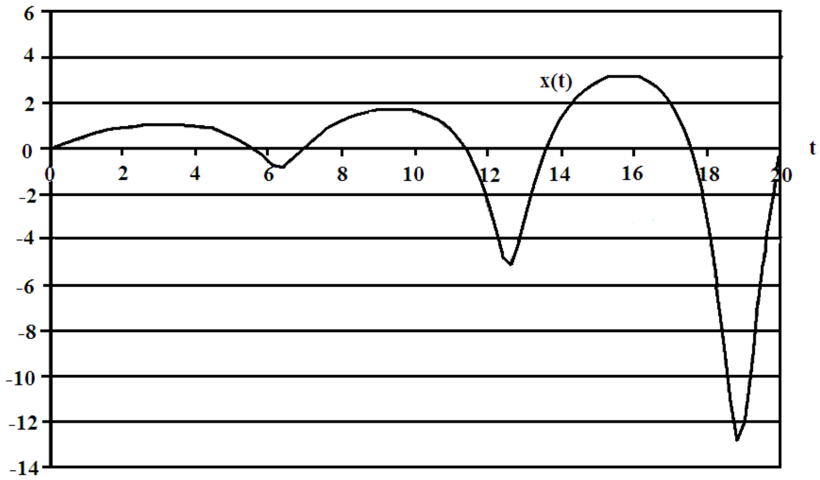

Figure 12 The shift function by the Rayleigh distribution of time to failure and the sinusoidal load rate

of the residual lifetime conservation to reliability analysis. The condition means that the cumulative distribution function of time to failure is continuous, i.e., it can not have jumps. It adequately models the reliability behavior of many real systems especially when we analyze continuous changes of load. Moreover, a natural assumption has been accepted in the paper. According to it, the change of the load in k times leads to change of the system failure rate in k times. By accepting the condition of the residual lifetime conservation and the rule for changing the failure rate, the problem of computing the reliability function taking into account changes of load is reduced to searching for the introduced time-dependent shift function $x(t)$ as a solution to the differential equation (5). Hence, the reliability function under conditions of the changed load is computed from (6).

For some special cases, including the case of the exponential probability distribution of time to failure, rather simple explicit expressions have been obtained for computing the reliability measures by various types of the load conditions. Generally, only numerical solutions can be obtained for arbitrary probability distributions of time to failure because we have to solve the differential equation (5) whose explicit solution can be found only for the special case of the exponential distribution. Nevertheless, the equation (5) can be solved by means of the well-known Runge-Kutta method. Moreover, the reliability function and the mean time to failure can be computed by using the corresponding algorithm.

The various numerical examples have illustrated how the different conditions of system functioning can impact on its reliability. In particular, the examples have shown that the form of the reliability function $P_c(t)$ as well as the reliability function $P(t)$ strongly depends on the mean square deviation. The largest time to failure under the same conditions takes place for a system with the exponentially distributed time to failure.

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