Probabilities of judgments provided by unknown experts by using the imprecise Dirichlet model

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Abstract

Most models of aggregating expert judgments assume that there is available some information characterizing the experts. This information may be incorporated into the so-called hierarchical uncertainty models (second-order models). However, we often do not know anything about experts or it is difficult to evaluate their quality. In this case, beliefs to experts may be in the interval [0,1] and the resulting assessments become to be non-informative. Moreover, attempts to assign some weights or beliefs to experts were not crowned with success because the behavior of experts may be distinguished in different circumstances. Therefore, this paper proposes to estimate expert judgments instead of experts themselves and studies how to assign interval probabilities of expert judgments by using a set of multinomial models.

Keywords: expert judgments, imprecise probabilities, hierarchical uncertainty, linear programming, multinomial model, Dirichlet distribution.

1 Introduction

Judgments elicited from human experts may be a very important part of information about systems on which limited experimental observations are possible. Several methods for elicitation, assessment and pooling of this type of information have been proposed in [1, 5, 13]. In order to get useful information from the experts, a proper uncertainty modeling of pieces of data supplied by experts has to be used. Uncertainty models play a central role in the use of expert judgments, because no human being would claim that he is absolutely sure about his judgments or advice [11, 21].

As indicated by Troffaes and de Cooman [24], there are two ways to approach the problem of aggregating expert opinions: *axiomatic* and *ad hoc*. Axiomatic approaches aim at deriving a preferably unique rule of aggregation from axioms or properties that this rule should satisfy. Ad hoc approaches are not as much concerned with axioms: one simply proposes or derives a mathematical formula, together with some form of justification. Both approaches have shortcomings and virtues, but axiomatic ones can be justified for various applications and initial data, whereas ad hoc approaches depend on specific applications and data.

Judgments elicited from experts are usually imprecise and unreliable due to the limited precision of human assessments. When several experts supply judgments or assessments about a system, their responses are pooled so as to derive a single measure of the system behavior. Judgments of reliable experts should be more important than those of unreliable ones. Various methods of the pooling of assessments, taking into account the quality of experts, are available in the literature [5, 13, 17, 18, 37]. These methods use the concept of precise probabilities for modelling the uncertainty and the quality of experts is modelled by means of *weights* assigned to every expert in accordance to some rules. It should be noted that most of these rules use some available information about correctness of previous expert opinions. This way might meet several difficulties. First, the behavior of experts is unstable, i.e., "exact" judgments related to a system elicited from an expert do not mean that this expert will provide results of the same quality for new systems. Second, when experts provide imprecise values of an evaluated quantity, the weighted rules can lead to controversial results. For instance, if an expert with a small weight, say 0.1, provides a very large interval, say $[0, 10]$, for a quantity (covering its sample space), it is obvious that this expert is too cautious and the interval he supplies is non-informative, although this interval covers a "true" value of the quantity. On the other hand, if an expert with a large weight, say 0.9, supplies a very narrow interval, say $[5, 5.01]$, the probability that "true" value of the quantity lies in this interval is rather small. We can see that the values of weights contradict with the probabilities of provided intervals. It should be noted that sometimes we do not know anything about quality of experts or assignment of weights meets some ethical difficulties. This implies that weights of experts as measures of their quality can not be measures of the quality of provided opinions.

Models of aggregating expert judgments taking into account the quality of experts can be considered in a framework of *hierarchical uncertainty models* which are rather common in uncertainty theory. Different application examples and a comprehensive review of hierarchical models can be found in [8, 9]. Hierarchical uncertainty models for aggregating expert judgment based on the imprecise probabilities have been considered in [15, 24, 25, 26, 27, 28, 29]. These models assume the lack of information about probability distributions on the first and second levels of the hierarchy. However, there is a difficulty in usage of the models. This is how to find the second-order probabilities or to estimate the quality of experts. Therefore, an approach for calculating the second-order probabilities of expert judgments is proposed in the paper.

2 Homogeneous judgments

2.1 Motivation

The main aim of the paper is to develop an approach for computing *second-order* probabilities of expert judgments and for aggregating expert opinions taking into account these probabilities. At that these probabilities are not regarded as a result of the previous expert experience, but as a result of "recent" judgments provided by unknown experts. The experts are unknown in the sense that we have no prior information about their quality.

What do the second-order probabilities mean? Several interpretations of the probabilities exist [8, 9, 19, 24, 33, 34]. In our case the second-order probabilities can measure our prior beliefs that the given interval-valued judgments cover some unknown "true" value of the corresponding statistical characteristics. It should be noted that many authors assume that the second-order probabilities are *subjective* ones given by some subject called the *modeller* [24] who is over *experts* providing the first-order information. The model studied here is quite different. It considers *objective* second-order probabilities obtained as a result of statistical inference on the basis of the expert first-order judgments.

A set of expert judgments jointly with their second-order probabilities are formally defined by a hierarchical model. A comprehensive review of hierarchical models is given in [8] where it is argued that the most common hierarchical model is the Bayesian one [2]. The Bayesian hierarchical model has been studied extensively and it has been found useful in many applications, notably those in which the subject can be assumed to be Bayesian and the modeller has sufficient information about the subject's probabilities to justify a precise second-order probability model. However, the Bayesian hierarchical model is unrealistic in problems where there is available only partial information about the system behavior or the number of experts is rather small. Moreover, the modeller is absent in my model and no prior information is used to evaluate the experts.

One of the promising tools for processing expert judgments is *imprecise* or *interval-valued probabilities* [16, 31, 35]. Coolen [7] indicated that the most important advantage of imprecise probability methods is that they do not require the expert to quantify his judgments via single numbers, explicitly allowing indeterminacy which could possibly reflect the confidence that the expert has in his knowledge with regard to the particular quantity of interest. Two main approaches for aggregating the expert judgments in the framework of imprecise probability theory can be pointed out. The first approach is the so-called *conjunction rule* [24] defined as the smallest lower expectation of the considered random quantity. Conjunction aims at gaining as much information as possible from each of the experts. It is realized by means of the *natural extension* which can be viewed as the linear optimization problem. Suppose a continuous random variable $X(x)$ is defined on a sample space Ω and the expert judgments about this variable are represented as a set of m interval-valued expectations $\underline{\mathbb{E}}f_i$ and $\overline{\mathbb{E}}f_i$, $i = 1, ..., m$, of functions $f_1(X), ..., f_m(X)$, respectively. For example, if f_i is the indicator function of an event A, then expectations $\underline{\mathbb{E}}f_i$ and $\overline{\mathbb{E}}f_i$ can be regarded as lower and upper probabilities of the event A. If $f_i(X) = X$, then $\underline{\mathbb{E}}f_i$ and $\overline{\mathbb{E}}f_i$ are bounds for the mean value of the corresponding random variable. In terms of imprecise probability theory the corresponding functions $f_i(X)$ and their interval-valued expectations are called *gambles* and *lower and upper previsions*, respectively. For computing new expectations $\underline{\mathbb{E}}g$ and $\overline{\mathbb{E}}g$ of a gamble $g(X)$ from the available information, the natural extension can be written as the following optimization problems:

$$\underline{\mathbb{E}}g = \min_{\pi} \int_{\Omega} g(x)\pi(x)\mathrm{d}x, \ \overline{\mathbb{E}}g = \max_{\pi} \int_{\Omega} g(x)\pi(x)\mathrm{d}x, \tag{1}$$

subject to

$$\pi(x) \geq 0, \ \int_{\Omega} \pi(x)\mathrm{d}x = 1, \ \underline{\mathbb{E}}f_i \leq \int_{\Omega} f_i(x)\pi(x)\mathrm{d}x \leq \overline{\mathbb{E}}f_i, \ i \leq m. \tag{2}$$

Here the minimum and maximum are taken over a set of all possible probability density functions $\{\pi(x)\}$ satisfying conditions (2). It should be noted that there are different representations (see [30]) of problems (1)-(2).

The above conjunction rule has a shortcoming. To show that we consider several examples. Let us imagine that two experts provide the judgments about the mean time to failure (MTTF) of a component: (1) MTTF is not greater than 10 hours; (2) MTTF is not less than 10 hours. The natural extension or conjunction rule produces the resulting MTTF $[0, 10] \cap [10, \infty) = 10$. In other words, the absolutely precise MTTF is obtained from extremely imprecise initial data. This is unrealistic in practice of reliability analysis. The reason of such results is that probabilities of judgments in the conjunction rule are assumed to be 1. If we assign some different probabilities to judgments, then we obtain more realistic assessments. For example, if beliefs to each judgment are 0.5, then, according to [14], the resulting MTTF is greater than 5 hours.

Let us consider another example. Suppose that many experts, say 1000, provide the same interval for some probability of failure, say $[0.8, 0.9]$ and one expert provides the interval $[0.6, 0.7]$. On one hand, the above judgments are conflicting and a set of probability distributions produced by these intervals is empty. As a result, we can not use the natural extension. The second approach for aggregating the expert judgments is the so-called *unanimity rule* defined as the envelope of the experts previsions [24], which is guaranteed to exist, but leads to extremely imprecise results (in the considered example, the resulting interval is $[0.6, 0.9]$). On the other hand, it is intuitively obvious that our belief to the judgment supplied by the last expert is rather low in comparison with our belief to the judgment provided by 1000 experts and the "unreliable" judgment could be removed from consideration. One might say that this example is highly artificial. Of course, the example is given here only for illustration purposes. However, what to do if only 2 experts instead of 1000 ones provide the interval $[0.8, 0.9]$ and one expert provides the interval $[0.6, 0.7]$. In this case, it is difficult to remove the contradictory interval. Moreover, it is difficult to determine what intervals are contradictory.

How can we assign probabilities to judgments if, for instance, we do not know anything about experts? To answer this question we consider the following example. Suppose that two experts supply two intervals, say $[4, 6]$ and $[5, 10]$ for the mean value of a random quantity defined on the sample space $\Omega = [0, 20]$. On one hand, by applying Walley's natural extension [31] to these judgments, we obtain the resulting interval of the mean value $[5, 6]$. On the other hand, one can conclude that the interval $[5, 6]$ has a larger probability than, for instance, intervals $[4, 5]$, $[6, 10]$, etc. Moreover, we do not know anything about intervals $[0, 4]$ and $[10, 20]$. This implies that we can assign some probabilities to every interval from Ω based only on the available information in the form of expert judgments.

What are conditions for these (second-order) probabilities? First, the probabilities have to take into account the incompleteness of the available information and even total ignorance. Second, the probabilities have to take into account the overcautiousness of experts when they supply too large and non-informative intervals. Third, the probabilities have to take into account the overconfidence of experts when they supply intervals that are too narrow (or point-values) [11, 12]. Fourth, the probabilities have to be simply updated after obtaining new judgments. Fifth, the probabilities are assigned not to experts, but to intervals provided by the experts. The first, second, and third conditions can be satisfied if to use imprecise probabilities. The fourth and fifth conditions are fulfilled if to assume that probabilities of intervals are governed by the Dirichlet distribution. Therefore, in order to satisfy all five conditions the imprecise Dirichlet model [32] is proposed to be used.

2.2 Basic idea

Let us consider an example of the *standard multinomial model*. Suppose there are L small identical boxes of size 1 located closely one after the other. One of the boxes contains a prize. Experts, say m ones, try to guess this box and to put a ball into the box with the prize. The first expert takes an oblong box of size l_1 with one open side. This box contains one ball which can move inside the box and we do not know location of the ball because the open side is behind. Then the expert covers l_1 small boxes by the oblong box and the ball enters in one of l_1 small boxes with numbers from a set J_1. At that the expert supposes that the ball must be in one of the chosen small boxes. We do not know exact location of the ball, but we know that it is in one of the boxes with numbers from J_1. Then the second expert takes another oblong box of size l_2 with an open side. The expert covers l_2 small boxes by the oblong box and the second ball enter in one of l_2 small boxes with numbers from a set J_2. This procedure is repeated m times. What can we say about balls in the small boxes now? Consider the following example. Let $m = 2$, $l_1 = 2$, $l_2 = 3$, $L = 5$, $J_1 = \{1, 2\}$, $J_2 = \{2, 3, 4\}$. Then the possible numbers of balls in 5 small boxes are

$$(1, 1, 0, 0, 0)$$
$$(1, 0, 1, 0, 0)$$
$$(1, 0, 0, 1, 0)$$
$$(0, 2, 0, 0, 0)$$
$$(0, 1, 1, 0, 0)$$
$$(0, 1, 0, 1, 0).$$

Let us study in detail the first possible allocation of two balls. The set $\Theta = \{1, 2, ..., L\}$ of all small boxes (their numbers) can be regarded as an exhaustive set of categories. If to assume that the experts are *independent*, then 2 observations are independently chosen from Θ, that is, two small boxes are chosen to find the prize with some unknown probabilities. This is the standard multinomial model. However, there exist different allocations of balls. Since we do not know anything about experts, then all allocations are equivalent in the sense that we can not choose one of them as a more preferable case. Therefore, we have a set of 6 equivalent multinomial models.

It is worth noticing that if we are not interested in studying every small box and consider only oblong boxes, then the set of categories can be reduced. For instance, we can unite third and fourth small boxes in the considered example and get the following non-redundant models with $L = 4$:

$$(1, 1, 0, 0)$$
$$(1, 0, 1, 0)$$
$$(0, 2, 0, 0)$$
$$(0, 1, 1, 0).$$

Suppose that the number of possible models is M. So, we can write all possible vectors of observations $\mathbf{n}^{(k)} = (n_1^{(k)}, ..., n_4^{(k)})$, $k = 1, ..., M$, for the multinomial models. For every model, the probability of an arbitrary event $A \subseteq \Theta$ depends on $\mathbf{n}^{(k)}$, that is, we can find $P(A|\mathbf{n}^{(k)})$. So far as all models are equivalent, even by precise probabilities of all categories only lower and upper probabilities of A can be computed

$$\underline{P}(A) = \min_{k=1,...,M} P(A|\mathbf{n}^{(k)}), \quad \overline{P}(A) = \max_{k=1,...,M} P(A|\mathbf{n}^{(k)}). \tag{3}$$

In particular, if all sets J_i consist of single components, that is, all oblong boxes are of size 1, then $M = 1$ and

$$\underline{P}(A) = P(A|\mathbf{n}^{(k)}), \quad \overline{P}(A) = P(A|\mathbf{n}^{(k)}).$$

The following problem is to define $P(A|\mathbf{n}^{(k)})$. In the case of multinomial samples, the Dirichlet distribution is the traditional choice[1].

2.3 Imprecise Dirichlet model

The *Dirichlet* (s,α) *prior distribution* for π, where $\alpha = (\alpha_1, ..., \alpha_m)$, has probability density function [36]

$$p(\pi) = C(s,\alpha) \cdot \prod_{j=1}^{m} \pi_j^{s\alpha_j - 1},$$

where $s > 0$, $0 < \alpha_j < 1$ for $j = 1, ..., m$, $\alpha \in S(1,m)$, and the proportionality constant C is determined by the fact that the integral of $p(\pi)$ over the simplex of possible values of π is 1 and

$$C(s,\alpha) = \Gamma(s) \left(\prod_{j=1}^{m} \Gamma(s\alpha_j) \right)^{-1}.$$

Here α_i is the mean of π_i under the Dirichlet prior and s determines the influence of the prior distribution on posterior probabilities. $\Gamma(\cdot)$ is the Gamma-function which satisfies $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(1) = 1$. $S(1,m)$ denotes the interior of the unit simplex.

Walley [32] pointed out several reasons for using a set of Dirichlet distributions to model prior ignorance about probabilities of states of nature:

1. Dirichlet prior distributions are mathematically tractable because they generate Dirichlet posterior distributions;

2. sets of Dirichlet distributions are very rich, because they produce the same inferences as their convex hulls and any prior distribution can be approximated by a finite mixture of Dirichlet distributions;

3. the most common Bayesian models for prior ignorance about probabilities of states of nature are Dirichlet distributions.

The *imprecise Dirichlet model* is defined by Walley [32] as the set of all Dirichlet (s,α) distributions such that $\alpha \in S(1,m)$.

For the imprecise Dirichlet model, the *hyperparameter* s determines how quickly upper and lower probabilities of events converge as statistical data accumulate. Walley [32] defined s as a number of observations needed to reduce the imprecision (difference between upper and lower probabilities) to half its initial value. Smaller values of s produce faster convergence and stronger conclusions, whereas large values of s produce more cautious inferences. At the same time, the value of s must not depend on m or a number of observations. The detailed discussion concerning the parameter s and the imprecise Dirichlet model can be found in [3, 6, 32]. The application of the model in reliability was studied in [6]. This model was also applied to the game theory for choosing a strategy in a two-player game by Quaeghebeur and de Cooman [20]. An approach

[1] It is worth noticing that the Dirichlet model should be regarded as one of the possible multinomial models that can be applied to the considered approach.

to dealing with incomplete sets of multivariate categorical data by exploiting Walley's imprecise Dirichlet model was studied by Zaffalon [38].

By returning to the multinomial models considered in the example with boxes and balls and assuming that small boxes are chosen to find the prize with probabilities governed by the Dirichlet distribution, we can write the lower and upper probabilities of an event A covering small boxes with numbers from the set J as

$$\underline{P}(A)=\min_{k=1,...,M}\inf_{\alpha\in S(1,L)}\frac{n^{(k)}(A)+s\alpha(A)}{m+s},$$

$$\overline{P}(A)=\max_{k=1,...,M}\sup_{\alpha\in S(1,L)}\frac{n^{(k)}(A)+s\alpha(A)}{m+s},$$

where

$$\alpha(A)=\sum_{j\in J}\alpha_j,\ \ n^{(k)}(A)=\sum_{j\in J}n_j^{(k)}.$$

2.4 Analysis of expert judgments

Let us reformulate the above example in terms of expert judgments and define how to find the set of minimal (non-redundant) "small boxes". Suppose that m experts provide m intervals $A_1=[\underline{a}_1,\overline{a}_1]$, $A_2=[\underline{a}_2,\overline{a}_2]$, ..., $A_m=[\underline{a}_m,\overline{a}_m]$ of the expectation[2] $\mathbb{E}Y$ of some function $Y=f(X)$ of a random quantity X such that Y is defined on a sample space[3] $\Omega=[\underline{A},\overline{A}]$. Define a set $\mathcal{B}$ of non-intersecting intervals B_i as follows. Let $\mathbf{i}_k=(i_1,...,i_m)$ be the k-th binary vector consisting of m components such that $i_j\in\{0,1\}$ and $k=1,...,M$, where $M=2^m$. For every vector $\mathbf{i}_k$, we determine B_k as intersection of intervals A_j whose indices correspond to non-zero components of $\mathbf{i}_k$ and of complements A_j^c whose indices correspond to zero components of $\mathbf{i}_k$, i.e.,

$$B_k=\left(\bigcap_{j:i_j=1}A_j\right)\bigcap\left(\bigcap_{j:i_j=0}A_j^c\right),\ i_j\in\mathbf{i}_k,\ k=1,...,2^m. \tag{4}$$

Then the set $\mathcal{B}$ is defined as a set of all non-identical intervals B_k. For example, if we have two intervals $A_1=[4,6]$, $A_2=[5,10]$ and $\Omega=[0,20]$, then $B_1=[0,4]\cup[10,20]$, $B_2=[4,5]$, $B_3=[6,10]$, $B_4=[5,6]$.

Suppose that the set $\mathcal{B}$ contains L intervals. Now every interval A_i can be represented as a union of l_i successive intervals $B_j\cup...\cup B_{j+l_i}$, $j\in\{1,...,L\}$. Denote $J_i=\{j,j+1,...,j+l_i\}$. Then $B_j\subseteq A_i$ if $j\in J_i$ and $A_i=\bigcup_{j\in J_i}B_j$.

If we associate intervals A_i with the oblong boxes provided by experts and B_j with united small boxes, then we can say that there is a set of multinomial models. Let the multinomial models have parameter vector $\pi^{(k)}=(\pi_1^{(k)},...,\pi_L^{(k)})$ with $\sum_{j=1}^L\pi_j^{(k)}=1$ and $0<\pi_j^{(k)}<1$. Then there holds

$$\Pr\{\mathbb{E}Y\in B_j\mid\pi^{(k)}\}=\pi_j^{(k)},\ j=1,...,L,\ k=1,...,M.$$

[2]Generally, the proposed method for analyzing expert judgements works with expert judgements related to some random variable X itself (not its expectation) in the same way. However, more often experts provide judgements about characteristics of random variables, e.g., probabilities, expectations, moments. Therefore, expectations of functions of X are considered here.

[3]The sample space Ω may be discrete. All expressions for probabilities of events in this case are the same.

Hence

$$\Pr\{\mathbb{E}Y \in A_i = \bigcup_{j\in J_i} B_j |\ \pi^{(k)}\} = \sum_{j\in J_i} \pi_j^{(k)},\ i=1,...,m,\ k=1,...,M.$$

So, the probability of an interval provided by an expert can be found as a sum of parameters of the Dirichlet distribution. Now a question is how to find $\pi^{(k)}$, $k=1,...,M$.

Let us explain what the interval A_i provided by the i-th expert means under condition $A_i = \bigcup_{j\in J_i} B_j$ (compare with the example about small and oblong boxes). By giving an opinion in the form of an interval, the expert supposes that the "true" value of $\mathbb{E}Y$ lies in this interval. This implies that the "true" value of $\mathbb{E}Y$ lies in one of the intervals $B_j, B_{j+1}, ..., B_{j+l_i}$, but if $l_i \neq 0$ or $J_i \neq \{\varnothing\}$, we do not know in what interval the value of $\mathbb{E}Y$ is. So, by obtaining the interval A_i, we have some event or a "trial", but we do not know its exact location in A_i and every interval B_j, $j \in J_i$, is possible, that is, there exist M possibilities. For instance, if $A_i = [2,4]$ and $B_j = [2,3]$, $B_{j+1} = [3,4]$, then, by denoting the number of trials in B_j by $n_j^{(k)}$, we have two possibilities ($M=2$): $n_j^{(1)} = 1$, $n_{j+1}^{(1)} = 0$ and $n_j^{(2)} = 0$, $n_{j+1}^{(2)} = 1$. Since every interval B_j is the intersection of a finite number of intervals A_i, $i \in \{1,...,m\}$, then for any $j \leq M$, $n_j^{(k)}$ takes values from the number of intervals $A_i \subseteq B_j$ to the number of intervals A_i covering B_j. For example, $A_i = [2,4]$, $A_l = [3,5]$ and $B_j = [3,4]$, then $n_j^{(1)} = 0$, or $n_j^{(2)} = 1$, or $n_j^{(3)} = 2$. Since we do not know exact values of $n_j^{(k)}$, then it is possible to find only lower and upper probabilities of events A_i. At that the lower (upper) probability of A_i is defined by such values of $n_j^{(k)}$, $j \in J_i$, that give the minimum (maximum) to the sum $\sum_{j\in J_i} \pi_j^{(k)}$. In other words, there hold

$$\underline{P}(A_i) = \min_{k=1,...,M} \Pr\{\mathbb{E}Y \in A_i | \mathbf{n}^{(k)}\} = \min_{k=1,...,M} \sum_{j:B_j\subseteq A_i} P(B_j|\mathbf{n}^{(k)}) = \min_{k=1,...,M} \sum_{j\in J_i} \pi_j^{(k)},$$

$$\overline{P}(A_i) = \max_{k=1,...,M} \Pr\{\mathbb{E}Y \in A_i | \mathbf{n}^{(k)}\} = \max_{k=1,...,M} \sum_{j:B_j\subseteq A_i} P(B_j|\mathbf{n}^{(k)}) = \max_{k=1,...,M} \sum_{j\in J_i} \pi_j^{(k)}.$$

According to the imprecise Dirichlet model, we get

$$\underline{P}(A_i) = \min_{k=1,...,M} \frac{n^{(k)}(A_i) + s\alpha(A_i)}{m+s},\ \overline{P}(A_i) = \max_{k=1,...,M} \frac{n^{(k)}(A_i) + s\alpha(A_i)}{m+s}, \tag{5}$$

Here $n^{(k)}(A_i) = \sum_{j:A_i\supseteq B_j} n_j^{(k)}$, $\alpha(A_i) = \sum_{j:A_i\supseteq B_j} \alpha_j$.

Denote

$$L_1(i) = \sum_{A_j:A_j\subseteq A_i} 1,\ L_2(i) = m - \sum_{A_j:A_i\subseteq A_j^c} 1, \tag{6}$$

where $L_1(i)$ is a number of intervals lying in A_i; $L_2(i)$ is a number of intervals whose intersection with A_i is empty.

Remark 1 *If an interval A_j is supplied by several experts, say n_j, i.e., there are n_j identical intervals, then $L_1(i)$ and $L_2(i)$ can be written as*

$$L_1(i) = \sum_{j:A_j^*\subseteq A_i} n_j,\ L_2(i) = m - \sum_{j:A_i^*\subseteq A_j^c} n_j,$$

where A_j^* are non-identical intervals.

Note that

$$L_1(i) = \min_{k=1,\dots,M} n^{(k)}(A_i),\ L_2(i) = \max_{k=1,\dots,M} n^{(k)}(A_i).$$

Then (5) can be rewritten as follows:

$$\underline{P}(A_i) = \frac{L_1(i) + s\alpha(A_i)}{m+s},\ \overline{P}(A_i) = \frac{L_2(i) + s\alpha(A_i)}{m+s}. \tag{7}$$

Finally, by applying the imprecise Dirichlet model, we can find the lower and upper probabilities of events A_i by solving the following optimization problems:

$$\underline{P}(A_i) = \inf_{\alpha \in S(1,L)} \frac{L_1(i) + s\alpha(A_i)}{m+s},\ \overline{P}(A_i) = \sup_{\alpha \in S(1,L)} \frac{L_2(i) + s\alpha(A_i)}{m+s}, \tag{8}$$

subject to $\alpha \in S(1,L)$.

It is interesting to note from (7) and (8) that we do not need to know intervals B_k. Indeed, $\inf_{\alpha \in S(1,L)} \underline{P}(A_i)$ is achieved at $\alpha(A_i) = 0$ and $\sup_{\alpha \in S(1,L)} \overline{P}(A_i)$ is achieved at $\alpha(A_i) = 1$ except a case when there is a non-informative interval $A_i = \Omega$. If $A_i = \Omega$, then $\alpha(A_i) = 1$ for the minimum and maximum. Therefore, we can write very simple expressions for computing lower and upper probabilities of intervals:

$$\underline{P}(A_i) = \frac{L_1(i)}{m+s},\ \overline{P}(A_i) = \frac{L_2(i) + s}{m+s}. \tag{9}$$

Remark 2 *It is noteworthy that the obtained results are very close to basic definitions of* random set theory *[10, 22]. Suppose that $\mathcal{P}(\Omega)$ is the power set of Ω. Then a function $\mu : \mathcal{P}(\Omega) \to [0,1]$, called basic probability assignment, is defined as*

$$\mu(\varnothing) = 1,\ \sum_{A \in \mathcal{P}(\Omega)} \mu(A) = 1.$$

According to [10], this function can be obtained as follows. Let c_i denote the number of occurrences of the set $A_i \subseteq \Omega$ from N observations. Then $\mu(A_i) = c_i/N$. According to [22], the belief $Bel(E)$ and plausibility $Pl(E)$ measures of an event $E \subseteq \Omega$ can be defined as

$$Bel(E) = \sum_{A_i : A_i \subseteq E} \mu(A_i),\ Pl(E) = \sum_{A_i : A_i \cap E \neq \varnothing} \mu(A_i). \tag{10}$$

These measures can be regarded as lower and upper bounds for the probability of E, i.e., $Bel(E) \leq \Pr(E) \leq Pl(E)$.

Now let us look at expressions (6) and (9). $L_1(i)$ and $L_2(i)$ are none other than $m \cdot Bel(A_i)$ and $m \cdot Pl(A_i)$. Here m is the number of experts or observations. Hence the lower and upper probabilities of A_i can be written through the belief and plausibility measures as

$$\underline{P}(A_i) = \frac{m \cdot Bel(A_i)}{m+s},\ \overline{P}(A_i) = \frac{m \cdot Pl(A_i) + s}{m+s}.$$

At that, it is obvious that $\underline{P}(A_i) \leq Bel(A_i)$ and $\overline{P}(A_i) \geq Pl(A_i)$. In particular, if $s = 0$ or $m \to \infty$, then $\underline{P}(A_i) = Bel(A_i)$ and $\overline{P}(A_i) = Pl(A_i)$. This is a very interesting fact which shows that the imprecise Dirichlet model can be expressed through belief and plausibility measures and can be regarded as some extension of these measures.

2.5 Aggregation of expert judgments

By using the natural extension, we can compute expectations $\underline{\mathbb{E}}Y$, $\overline{\mathbb{E}}Y$ as follows:

$$\underline{\mathbb{E}}Y = \inf_{\mathcal{P}} \int_{\Omega} y\rho(y)\mathrm{d}y, \quad \overline{\mathbb{E}}Y = \sup_{\mathcal{P}} \int_{\Omega} y\rho(y)\mathrm{d}y,$$

subject to

$$\underline{P}(A_i) \leq \int_{\Omega} I_{A_i}(y)\rho(y)\mathrm{d}y \leq \overline{P}(A_i), \ i = 1, ..., m.$$

Here $\mathcal{P}$ is the set of all probability density functions ρ. This problem can be written in another form [16]. For example, the lower bound $\underline{\mathbb{E}}f$ is determined as:

$$\underline{\mathbb{E}}Y = \sup \left(c_0 + \sum_{i=1}^{m} c_i \underline{P}(A_i) - d_i \overline{P}(A_i) \right)$$

subject to $c_0 \in \mathbb{R}$, $c_i, d_i \in \mathbb{R}_+$, $i = 1, ..., m$, and

$$c_0 + \sum_{i=1}^{m} (c_i - d_i) I_{A_i}(y) \leq y, \ \forall y \in \Omega.$$

Example 1 *Suppose that there are three expert judgments about the mean value of a random variable, for instance, time to failure, defined on* $\Omega = [0, 20]$*. These judgments are given in the form of intervals* $A_1 = [3, 5]$*,* $A_2 = [1, 4]$*, and* $A_3 = [0, 5]$ *(see Fig.1). It is obvious from (6) and (9) that*

$$\underline{P}(A_1) = \frac{1}{3+s}, \ \overline{P}(A_1) = 1,$$

$$\underline{P}(A_2) = \frac{1}{3+s}, \ \overline{P}(A_2) = 1,$$

$$\underline{P}(A_3) = \frac{3}{3+s}, \ \overline{P}(A_3) = 1.$$

Suppose $s = 1$*. Then the aggregated interval for* $\mathbb{E}Y$ *can be found by means of the natural extension as*

$$\underline{\mathbb{E}}Y = \inf_{\mathcal{P}} \int_{\Omega} y\rho(y)\mathrm{d}y, \ \overline{\mathbb{E}}Y = \sup_{\mathcal{P}} \int_{\Omega} y\rho(y)\mathrm{d}y$$

subject to

$$\frac{1}{4} \leq \int_{\Omega} I_{A_1}(y)\rho(y)\mathrm{d}y \leq 1, \ \frac{1}{4} \leq \int_{\Omega} I_{A_2}(y)\rho(y)\mathrm{d}y \leq 1, \ \frac{3}{4} \leq \int_{\Omega} I_{A_3}(y)\rho(y)\mathrm{d}y \leq 1.$$

Hence $\underline{\mathbb{E}}Y = 0.75$*,* $\overline{\mathbb{E}}Y = 4.7$*. If we assume that* $s = 0$*, then* $\underline{\mathbb{E}}Y = 1$*,* $\overline{\mathbb{E}}Y = 4.6$*.*

Example 2 *Suppose that* 10 *experts supply intervals for probabilities of an event* A *(see Table 1). The intervals* A_i *are given in the second column. Numbers of identical intervals* n_i *are given in the third columns. Values of* $L_1(i)$ *and* $L_2(i)$ *calculated from (6) are given in the fourth and fifth columns. If we take* $s = 1$*, then* $\underline{P}(A_i)$ *and* $\overline{P}(A_i)$ *calculated from (9) are given in the sixth*

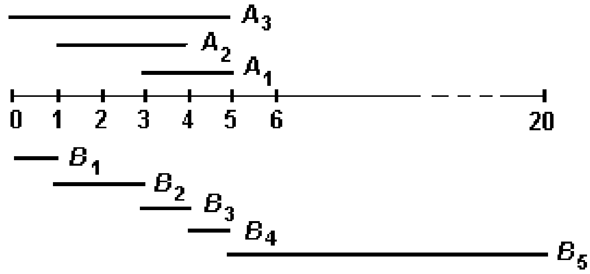

Figure 1: Intervals A_i provided by experts and produced intervals B_k

and seventh columns. The supplied intervals are depicted in Fig.2. Then the aggregated interval for $\mathbb{E}Y$ *(probability of the event* A*) can be found by means of the natural extension as*

$$\underline{\mathbb{E}}Y = \inf_{\mathcal{P}} \int_0^1 y\rho(y)\mathrm{d}y,\ \overline{\mathbb{E}}Y = \sup_{\mathcal{P}} \int_0^1 y\rho(y)\mathrm{d}y$$

subject to

$$0.45 \leq \int_0^1 I_{A_1}(y)\rho(y)\mathrm{d}y \leq 0.82,\ 0.73 \leq \int_0^1 I_{A_2}(y)\rho(y)\mathrm{d}y \leq 1,$$

$$0.91 \leq \int_0^1 I_{A_3}(y)\rho(y)\mathrm{d}y \leq 1,\ 0.55 \leq \int_0^1 I_{A_4}(y)\rho(y)\mathrm{d}y \leq 0.82,$$

$$0.18 \leq \int_0^1 I_{A_5}(y)\rho(y)\mathrm{d}y \leq 0.82,\ 0.18 \leq \int_0^1 I_{A_6}(y)\rho(y)\mathrm{d}y \leq 0.45.$$

Hence $\underline{\mathbb{E}}Y = 0.16$*,* $\overline{\mathbb{E}}Y = 0.35$*.*

Let us compute lower and upper probability distributions over all intervals B_k *from (4). This can be carried out by means of the natural extension of probabilities of* A_i *on probabilities of* B_k*. The lower (curve 1) and upper (curve 2) probability distributions are shown in Fig.3.*

Let us illustrate how the probabilities of the supplied intervals are updated after obtaining new intervals. Suppose that four new experts provide intervals $[0.2, 0.25]$, $[0.2, 0.25]$, $[0.25, 0.3]$, $[0.2, 0.3]$. *Then Table 1 is rewritten in the form of Table 2 and there hold* $\underline{\mathbb{E}}Y = 0.17$*,* $\overline{\mathbb{E}}Y = 0.33$*.*

Example 3 *We consider an example of analyzing the interval values of the so-called rock mass rating (RMR) [4] in rock engineering (a 6-m radius tunnel must be driven in the host rock (limestone) of Masua Mine in Sardinia, Italy, at a depth of 350 m). The interval values* A_i *of RMR have been presented by Tonon, Bernardini, and Mammino [23] and are given in the second and third columns of Table 3. The lower* $\underline{P}(A_i)$ *and upper probabilities* $\overline{P}(A_i)$*, computed in accordance with the imprecise Dirichlet model, are shown in Table 3. According to [23], the set of values of RMR is* $[36.5, 81.5]$*. The aggregated interval for RMR by* $s = 1$ *can be found as follows:*

$$\underline{\mathbb{E}}Y = \inf_{\mathcal{P}} \int_{36.5}^{81.5} y\rho(y)\mathrm{d}y,\ \underline{\mathbb{E}}Y = \sup_{\mathcal{P}} \int_{36.5}^{81.5} y\rho(y)\mathrm{d}y$$

Table 1: Intervals supplied by experts and their lower and upper probabilities

i	A_i	n_i	$L_1(i)$	$L_2(i)$	$\underline{P}(A_i)$	$\overline{P}(A_i)$
1	$[0.2, 0.3]$	3	5	8	0.45	0.82
2	$[0.15, 0.3]$	1	8	10	0.73	1
3	$[0.1, 0.4]$	1	10	10	0.91	1
4	$[0.2, 0.4]$	1	6	8	0.55	0.82
5	$[0.2, 0.25]$	2	2	8	0.18	0.82
6	$[0.15, 0.2]$	2	2	4	0.18	0.45

Table 2: Intervals supplied by experts and updated lower and upper probabilities

i	A_i	n_i	$L_1(i)$	$L_2(i)$	$\underline{P}(A_i)$	$\overline{P}(A_i)$
1	$[0.2, 0.3]$	4	9	12	0.6	0.87
2	$[0.15, 0.3]$	1	12	14	0.8	1
3	$[0.1, 0.4]$	1	14	14	0.93	1
4	$[0.2, 0.4]$	1	10	12	0.67	0.87
5	$[0.2, 0.25]$	4	4	11	0.27	0.8
6	$[0.15, 0.2]$	2	2	4	0.13	0.33
7	$[0.25, 0.3]$	1	1	8	0.07	0.6

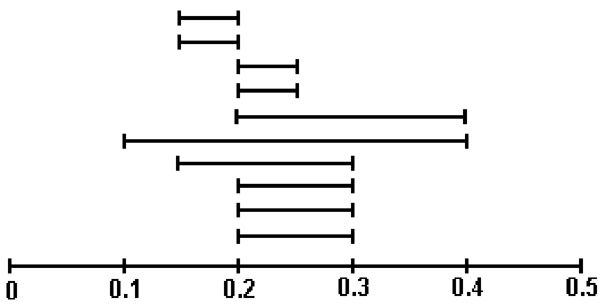

Figure 2: Intervals A_i provided by experts

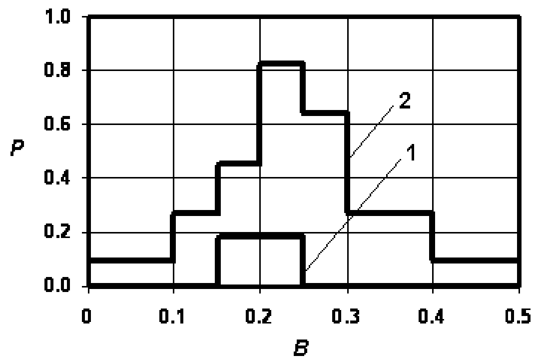

Figure 3: Lower and upper probability distribution functions

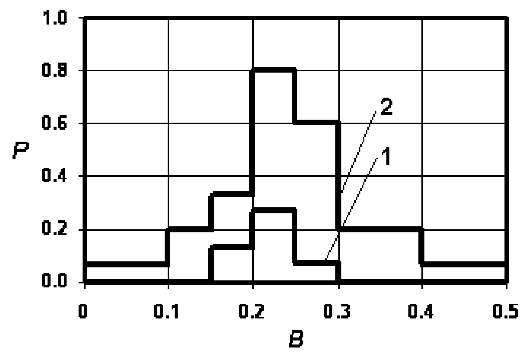

Figure 4: Updated lower and upper probability distribution functions

Table 3: Intervals of RMR and their probabilities

i	A_i	n_i	$L_1(i)$	$L_2(i)$	$\underline{P}(A_i)$	$\overline{P}(A_i)$
1	$[56, 74.5]$	2	2	30	0.0645	1
2	$[56, 81.5]$	1	3	30	0.0968	1
3	$[48.5, 69.5]$	9	9	30	0.2903	1
4	$[48.5, 76.5]$	2	11	30	0.3548	1
5	$[46.5, 69.5]$	1	10	30	0.3226	1
6	$[45.5, 63.5]$	1	1	30	0.0323	1
7	$[41, 62]$	3	3	30	0.0968	1
8	$[41, 69]$	1	5	30	0.1613	1
9	$[39, 62]$	9	12	30	0.3871	1
10	$[36.5, 62]$	1	13	30	0.4333	1

subject to

$$\underline{P}(A_i) \leq \int_{36.5}^{81.5} I_{A_i}(y)\rho(y)\mathrm{d}y \leq \overline{P}(A_i),\ \ i = 1, ..., 10.$$

Hence $\underline{\mathbb{E}}Y = 41.62$*,* $\overline{\mathbb{E}}Y = 73.02$*. It is necessary to note that mean values of RMR obtained in [23] are* $\underline{\mathbb{E}}^*Y = 44.83$*,* $\overline{\mathbb{E}}^*Y = 67.23$*.*

2.6 Special cases

Let us consider some important special cases.

1. *Coinciding judgments.* Suppose that there are m coinciding judgments, i.e., $A_1 = ... = A_m = [\underline{a}, \overline{a}]$. This means that all experts have the same opinions about some unknown value of $\mathbb{E}Y$. Then we can write $B_1 = [\inf \mathbb{E}Y, \underline{a}] \cup [\overline{a}, \sup \mathbb{E}Y]$, $B_2 = [\underline{a}, \overline{a}]$, $n(A_i) = m$, and

$$\underline{P}(A_i) = \frac{m}{m+s},\ \overline{P}(A_i) = 1.$$

If $m \to \infty$, then there holds $\underline{P}(A_i) = \overline{P}(A_i) = 1$. This property supports the intuitive sense. Indeed, if we have many identical judgments, we begin to think that these judgments are "true“ even if we do not know anything about each expert. If $m = 1$, then there hold

$$\underline{P}(A_i) = \frac{1}{1+s},\ \overline{P}(A_i) = 1.$$

The result corresponding to the case $m = 1$ shows that the precise Dirichlet model ($s = 0$) gives lower and upper probabilities of events 1. This is the incorrect conclusion. How can we totally rely on one judgment? This contradiction can be avoided by using the imprecise Dirichlet model ($s > 0$).

2. *Conflicting judgments.* Suppose that there are 2 conflicting judgments $A_1 = [\underline{a}_1, \overline{a}_1]$, $A_2 = [\underline{a}_2, \overline{a}_2]$ such that $\overline{a}_1 < \underline{a}_2$, i.e., $A_1 \cap A_2 = \varnothing$. Then we can write $B_1 = [\inf \mathbb{E}Y, \underline{a}_1] \cup$

$[\overline{a}_1, \sup \mathbb{E}Y] \cup [\overline{a}_1, \underline{a}_2]$, $B_2 = [\underline{a}_1, \overline{a}_1]$, $B_3 = [\underline{a}_2, \overline{a}_2]$. Hence

$$\underline{P}(A_i) = \frac{1}{2+s},\ \overline{P}(A_i) = \frac{1+s}{2+s}.$$

If there are m conflicting judgments, then

$$\underline{P}(A_i) = \frac{1}{m+s},\ \overline{P}(A_i) = \frac{1+s}{m+s}.$$

If $m \to \infty$, then there holds $\underline{P}(A_i) = \overline{P}(A_i) = 0$.

3. *Non-informative judgments (overcautiousness of experts).* Suppose that there is one judgment $A_1 = [\inf \mathbb{E}Y, \sup \mathbb{E}Y]$. Then $B_1 = A_1$, $\alpha(A_1) = 1$, and

$$\underline{P}(A_1) = \frac{1}{1+s},\ \overline{P}(A_1) = 1.$$

However, since $A_1^c = \varnothing$, then, by using the natural extension to avoid the redundancy of $\underline{P}(A_1)$ and $\overline{P}(A_1)$, we get $\underline{P}(A_1) = \overline{P}(A_1) = 1$. This implies that the overcautiousness of experts is properly modelled. Indeed, even if we do not believe that an expert provide the correct judgment, but he (she) provides very overcautious judgments, then probabilities of these judgments have to be large though the expert is unreliable.

3 Heterogeneous judgments

Let us study a case when experts provide their judgments different in kind. For example, the first expert gives an interval of a probability and the second one provides an interval of the expectation of a random variable. Suppose that we have m judgments of the type

$$\underline{\mathbb{E}}f_i \leq \mathbb{E}f_i \leq \overline{\mathbb{E}}f_i,\ i = 1, ..., m. \tag{11}$$

Here f_i is a function of the random variable X, $\underline{\mathbb{E}}f_i$ and $\overline{\mathbb{E}}f_i$ are lower and upper expectations of the function f_i. Our aim is to determine the lower and upper probabilities of these judgments.

Let us consider a special case of two judgments about a discrete random variable taking three values. These judgments induce two sets of probability distributions $\{p_1, p_2, p_3\}$ which are depicted in Fig.5 in the form of polyhedrons $ABCD$ and $EFGH$, respectively. By assuming the sample space $\Omega = S(1,3)$ (the set of all points in the simplex), we define events A_1 and A_2 as sets of all points in polyhedrons $ABCD$ and $EFGH$, respectively. Then we can define events B_k as follows: B_0 is the set of all points in polyhedrons $AKGR$, HLC, $DMFS$, $QBNE$; B_1 is the set of all points in polyhedrons $KGHL$, $NMEF$; B_2 is the set of all points in polyhedrons $BAKN$, $LCDM$; B_3 is the set of all points in the polyhedron $KLMN$. By assuming probabilities that any point taken from the simplex corresponds to events B_0, B_1, B_2, B_3 are governed by the imprecise Dirichlet distributions, we can find probabilities of events A_1 and A_2 in the same way as it has been done in the case of homogeneous judgments.

Now the following questions arise. How to find the events B_k in the general case? How to compute probabilities of A_i, $i = 1, ..., m$? This is similarly to the case of homogeneous intervals A_i. The difference is that we have polyhedrons instead of intervals, but every polyhedron can

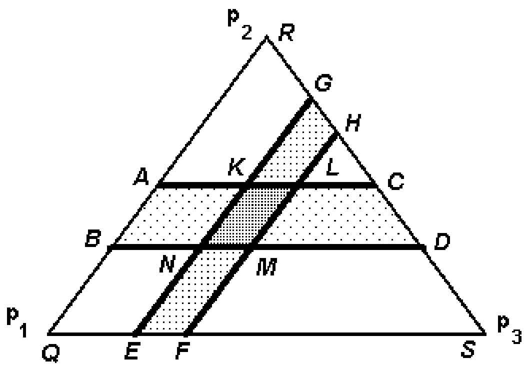

Figure 5: Illustration of probability sets corresponding to two judgements

be represented in the form of an interval. Therefore, we can use expressions (9) for computing probabilities of events (11). The main problem here is to define what conditions $A_k \subseteq A_i$ and $A_i \subseteq A_k^c$ mean. The first condition mean that a set $\mathcal{P}_k$ of probability distributions produced by the constraint $\underline{\mathbb{E}}f_k \leq \mathbb{E}f_k \leq \overline{\mathbb{E}}f_k$ is a part of a set $\mathcal{P}_i$ of probability distributions produced by the constraint $\underline{\mathbb{E}}f_i \leq \mathbb{E}f_i \leq \overline{\mathbb{E}}f_i$, i.e., $\mathcal{P}_k \subseteq \mathcal{P}_i$. The second condition can be rewritten as $A_i \cap A_k = \varnothing$, and this means that sets $\mathcal{P}_k$ and $\mathcal{P}_i$ produced by A_k and A_i, respectively, do not have common distributions, i.e., $\mathcal{P}_k \cap \mathcal{P}_i = \varnothing$.

Proposition 1 *Let* $C = [\underline{c}, \overline{c}]$ *be an interval such that* $\underline{c} = \inf_{\mathcal{P}_k} \mathbb{E}f_i$, $\overline{c} = \sup_{\mathcal{P}_k} \mathbb{E}f_i$. *Then* $\mathcal{P}_k \subseteq \mathcal{P}_i$ *if* $C \subseteq [\underline{\mathbb{E}}f_i, \overline{\mathbb{E}}f_i]$.

Proof. The proof is obvious and follows from the condition that the set $\mathcal{P}_k$ is a part of $\mathcal{P}_i$. ■

Proposition 1 implies that for satisfying the first condition $A_k \subseteq A_i$ we have to extend $\underline{\mathbb{E}}f_k$ and $\overline{\mathbb{E}}f_k$ on the bounds $\underline{c}$ and $\overline{c}$ of the linear expectation $\mathbb{E}f_i$ by means of the natural extension

$$\underline{c} = \inf_{\mathcal{P}} \mathbb{E}f_i, \ \overline{c} = \sup_{\mathcal{P}} \mathbb{E}f_i,$$

subject to $\underline{\mathbb{E}}f_k \leq \mathbb{E}f_k \leq \overline{\mathbb{E}}f_k$.

Here $\mathcal{P}$ is a set of all probability distributions.

The following aggregation of intervals can be carried out by methods described in [25, 26].

Proposition 2 *The second condition is satisfied if the linear programming problem with an arbitrary objective function and constraints*

$$\underline{\mathbb{E}}f_k \leq \mathbb{E}f_k \leq \overline{\mathbb{E}}f_k, \ \underline{\mathbb{E}}f_i \leq \mathbb{E}f_i \leq \overline{\mathbb{E}}f_i$$

does not have any solution. In this case, the first and the second constraints produce non-intersecting sets of probability distributions.

Proof. The proof is obvious. ■

Example 4 *Suppose that there are the following judgments about a discrete random variable* X *defined in the sample space* $\Omega = \{1,2,3\}$*: the mean value of* X *is between states* 1 *and* 2*; the probability that* $X = 3$ *is less than* 0.2*. This information is formally represented as*

$$1 \leq \mathbb{E}X \leq 2,\ 0 \leq \mathbb{E}I_{\{3\}}(X) \leq 0.3.$$

Let us find the lower and upper probabilities of these judgments. We have two events

$$A_1 = \{1 \leq \mathbb{E}X \leq 2\},\ A_2 = \{0 \leq \mathbb{E}I_{\{3\}}(X) \leq 0.3\}.$$

Note $A_2 \nsubseteq A_1$ *because the linear programming problems*

$$\inf(\sup)\mathbb{E}I_{\{3\}}(X) = \inf(\sup)(0p_1 + 0p_2 + 1p_3)$$

subject to

$$1 \leq 1p_1 + 2p_2 + 3p_3 \leq 2,$$

$$p_1 + p_2 + p_3 = 1,$$

has solutions $\underline{c} = 0$*,* $\overline{c} = 0.5$ *and* $[0, 0.3] \subset [0, 0.5]$*. Therefore, there holds* $L_1(1) = 1$*.*

Note $A_1 \nsubseteq A_2$ *because the linear programming problems*

$$\inf(\sup)\mathbb{E}X = \inf(\sup)(1p_1 + 2p_2 + 3p_3)$$

subject to

$$0 \leq p_3 \leq 0.3,\ p_1 + p_2 + p_3 = 1,$$

has solutions $\underline{c} = 1$*,* $\overline{c} = 2.3$ *and* $[1, 2] \subset [1, 2.3]$*. Therefore, there holds* $L_1(2) = 1$*. Note that given constraints are consistent because the optimization problem (we can take an arbitrary objective function)*

$$\inf(\sup)p_1$$

subject to

$$1 \leq 1p_1 + 2p_2 + 3p_3 \leq 2,$$

$$0 \leq p_3 \leq 0.3,\ p_1 + p_2 + p_3 = 1,$$

has a solution. Therefore, $L_2(i) = 2$*,* $i = 1, 2$*. Hence*

$$\underline{P}(A_i) = \frac{1}{2+s},\ \overline{P}(A_i) = \frac{2+s}{2+s} = 1,\ i = 1, 2.$$

Example 5 *Suppose that* m *experts provide judgments about quantiles of an unknown probability distribution* $p = (p_1, ..., p_n)$ *characterizing a discrete random variable* X *defined on the sample space* $\Omega = \{1, 2, ..., 20\}$*. According to [5], experts supply numbers* v *such that* $\sum_{j<v} p_j \geq q$*. Here* v *is the* q*-quantile of* p*. Let* $A_k^{(v)}$ *be the event corresponding to the* k*-th expert judgment about* q_k*-quantile and* $A_k^{(v)} = \{1, ..., v-1\}$*. Then with respect to the definition of the expert judgment about quantile, there holds* $q_k \leq \Pr\{A_k^{(v)}\} \leq 1$ *or*

$$q_k \leq \mathbb{E}I_{A_k^{(v)}}X \leq 1.$$

Table 4: Quantiles supplied by experts and their lower and upper probabilities

k	$q_k\%$	v	n_k	$L_1(k)$	$L_2(k)$	$\underline{P}(A_{k,v})$	$\overline{P}(A_{k,v})$
1	5 %	2	3	24	24	0.96	1
2	5 %	3	4	21	24	0.84	1
3	5 %	4	1	20	24	0.8	1
4	50 %	7	2	18	24	0.72	1
5	50 %	9	2	16	24	0.64	1
6	50 %	10	3	13	24	0.52	1
7	50 %	12	1	12	24	0.48	1
8	95 %	17	2	10	24	0.4	1
9	95 %	18	5	5	24	0.2	1
10	95 %	19	1	4	24	0.16	1

As far as the judgments are represented as expectations of different functions $I_{A_k^{(v)}}X$*, then these judgments can be regarded as heterogeneous ones and Propositions 1 and 2 can be applied to computing probabilities of judgments. Suppose* 28 *experts (*$m = 28$*) supply judgments about* 5%, 50%, 95% *quantiles. Among all judgments we can select* 10 *different ones. The numbers* v *supplied by experts (q-quantiles) are given in the third column of Table 4. Numbers of identical judgments (*n_k*) are given in the fourth column. Values of* $L_1(k)$ *and* $L_2(k)$ *calculated from (6) are given in the fifth and sixth columns. If we take* $s = 1$*, then* $\underline{P}(A_{k,v})$ *and* $\overline{P}(A_{k,v})$ *calculated from (9) are given in the seventh and eighth columns. Here the event* $A_{k,v}$ *means that* v *is the* q_k*-quantile, i.e.,* $A_{k,v} = \{q_k \leq \mathbb{E}I_{A_k^{(v)}}X \leq 1\}$*. It can be seen from Table 4 that all upper probabilities are* 1*. This is due to the lack of conflicting judgments in the example.*

4 Conclusion

This paper can be regarded as an attempt to find a way for constructing the hierarchical uncertainty model and for combining the interval-valued expert judgments without additional, sometimes erroneous, prior assumptions concerning probabilities or weights of experts. An approach for computing the second-order probabilities proposed in the paper also has the following additional advantages:

1. The obtained probabilities can be simply updated after observing new events or obtaining new expert judgments because Dirichlet prior distributions generate Dirichlet posterior distributions.

2. The obtained hierarchical uncertainty model is more realistic in many applications in the case of a small number of judgments, incomplete, and imprecise data. The possible large imprecision of results reflects insufficiency of available information.

3. The model allows us to make cautious inference about probabilities of judgments due to the controllable hyperparameter s of the imprecise Dirichlet model. As a result, the obtained probabilities of judgments take into account the incompleteness of the available information and total ignorance.

4. It can be seen from Subsection 2.6 (the third special case) that the overcautiousness of experts is properly modelled. The same can be seen in the examples. For instance, the third judgment in Example 1 and the tenth judgment in Example 3 are rather overcautious and their probabilities are maximal. The model takes into account the overconfidence of experts. It can be seen from Example 2 that the fifth and sixth judgments are overconfident. As a result, their probabilities are minimal. It is worth noticing that the overcautiousness and overconfidence of experts are automatically determined by computing the posterior probabilities of judgments.

5. The model allows us to deal with conflicting judgments without removing them.

Acknowledgement

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