A new ranking procedure by incomplete pairwise comparisons using preference subsets

Lev V. Utkin*
Department of Computer Science
St.Petersburg State Forest Technical Academy
Institutski per. 5, 194021, St.Petersburg, Russia
e-mail: lev.utkin@mail.ru

Abstract

A method for ranking of alternatives or objects and its extensions by incomplete pairwise comparisons using Dempster-Shafer theory are proposed in the paper. The main feature of the method is that it allows us to deal with comparisons of arbitrary subsets of alternatives. The method is extended on the case of independent groups of experts. The imprecise Dirichlet model is also used to make cautious decisions in several cases. Various numerical examples illustrate the proposed method and its extensions.

Keywords: expert judgments, pairwise comparison, preferences, Dempster-Shafer theory, belief and plausibility functions, imprecise Dirichlet model, ranking.

1 Introduction

Many application problems (multi-attribute decision making, data classification, etc.) deal with ranking of alternatives or objects. The *ranking* of n alternatives from the set $\mathbb{A} = \{A_1, ..., A_n\}$ can be expressed as[1]

$$A_{i_1} \succeq A_{i_2} \succeq ... \succeq A_{i_n}, \tag{1}$$

and means that the alternative A_{i_1} is preferred to A_{i_2}, the alternative A_{i_2} is preferred to A_{i_3}, ... , the alternative $A_{i_{n-1}}$ is preferred to A_{i_n}. We will denote the ranking of n alternatives by the sequence of indices $(i_1, ..., i_n)$ corresponding to (1).

There are a lot of ranking procedures depending on initial data and elicitation techniques. An interesting and comprehensive review of ranking procedures and their comparison have been carried out by Hüllermeier and Fürnkranz [9, 8]. An important class of elicitation techniques consists of the psychological scaling models [13] that use the concept of paired comparisons. Therefore, one of the prevailing ways for getting initial data for ranking is pairwise comparisons.

*Corresponding author.

[1] The ranking can also be expressed as $A_{i_1} \succ A_{i_2} \succ ... \succ A_{i_n}$. This means that the alternative A_{i_1} is strictly preferred to A_{i_2}, ... , the alternative $A_{i_{n-1}}$ is strictly preferred to A_{i_n}.

The popularity of the paired comparison methods can perhaps be contributed to the observation that experts are more comfortable making comparisons rather than directly assessing a quantity of interest [5].

There are various methods of pairwise comparisons. One of the well-known methods is the Analytic Hierarchy Process (AHP) [13] where experts supply the ratio of their preferences of one decision over another. However, in spite of possible simplifications of this method, this elicitation procedure may be rather difficult for experts sometimes when the number of alternatives is large, when experts are able to compare only subsets of alternatives and can not compare separate alternatives in these subsets, when experts can not give the degree of preferences for pairs of alternatives.

In order to overcome some difficulties, Beynon *et al* [3, 4] proposed a method using *Dempster-Shafer theory* (DST) and called the *DS/AHP method*. The method was developed for decision problems with a single decision maker, and it applies the AHP for collecting the preferences from a decision maker and for modelling the problem as a hierarchical decision tree. An extension of the method was proposed by Tervonen *et al* [16]. It should be noted that the main excellent idea underlying the DS/AHP method is not applying Dempster-Shafer theory to the AHP. It is the comparison of groups (subsets) of alternatives with a whole set of alternatives, i.e., instead of comparing alternatives between each other, the decision maker has to identify favorable alternatives from the set $\mathbb{A}$. This type of comparison is equivalent to the preference stated by the decision maker. In spite of many advantages of the DS/AHP method, it can not compare groups of alternatives. Suppose that there are a lot of transport facilities which can be divided into three groups: motor transport, air transport and water transport. An expert can not provide pairwise comparison of all the facilities, but he (she) can say that motor transport is more preferable than water transport. This expert does not supply estimates of separate transport facilities. Moreover, we do not know anything about air transport from the given expert judgment. How to process the above incomplete information? How to combine the incomplete information from many experts? How to take into account the possible situation when the number of experts is very small?

One of the simplest ways for dealing with the incomplete expert judgments is the following. If expert chooses a group of alternatives and can not distinguish separate alternatives in this group, then we could assume that all alternatives in the group are identical and they have uniformly distributed weights. However, this assumption is too strong. This way can lead to incorrect solutions because we introduce some additional information that we do not have.

It is necessary to point out that Beynon *et al* [3, 4] proposed to compare groups of alternatives by means of their separate comparisons with the set $\mathbb{A}$ and assignments different rates to the comparisons. However, it is difficult to assign a numerical value of the favorable opinion for a particular group of alternatives. Moreover, when several experts provide their judgments, it is difficult to get consistent estimates.

Therefore, a new ranking procedure is proposed in the paper. This procedure deals with the expert group pairwise comparisons and uses DST for processing the expert estimates. The main feature of the proposed ranking procedure is that experts compare not only single alternatives, but also arbitrary groups of alternatives. An algebra of sets of preferences with a certain operations is studied here.

Note that pairwise comparisons can be elicited from different independent sources. Therefore, the so-called Dempster's rule of combination for combining independent sources of data and its modifications in the framework of the sets of preferences are proposed. Another problem solved in the paper is to develop a method for implementing the cautious ranking procedures taking into account the possible small number of judgments. This approach is based on using the imprecise Dirichlet model [21] and on extension of belief and plausibility measures [18]. Various numerical examples illustrate the proposed approaches.

The paper is organized as follows. Some definitions and elements of DST are given in Section 2. The main idea of the proposed ranking method is considered in Section 3. In this section, the set-theoretic operations with preferences are introduced and it is shown how DST can be applied for processing the expert judgments. An extension of the proposed method on the case when comparisons are supplied by independent groups of experts is studied in Section 4. The cautious decisions with using the imprecise Dirichlet model, which have a number of advantages improving the proposed ranking method, are considered in Section 5.

2 Dempster-Shafer theory

Let U be a universal set under interest, usually referred to in evidence theory as the *frame of discernment*. Suppose N observations were made of an element $u \in U$, each of which resulted in an imprecise (non-specific) measurement given by a set A of values. Let c_i denote the number of occurrences of the set $A_i \subseteq U$, and $\mathcal{P}o(U)$ the set of all subsets of U (power set of U). A frequency function m, called *basic probability assignment* (BPA), can be defined such that [6, 15]:

$$m : \mathcal{P}o(U) \to [0,1],$$

$$m(\varnothing) = 1, \quad \sum_{A \in \mathcal{P}o(U)} m(A) = 1.$$

Note that the domain of BPA, $\mathcal{P}o(U)$, is different from the domain of a probability density function, which is U. According to [6], this function can be obtained as follows:

$$m(A_i) = c_i/N. \tag{2}$$

If $m(A_i) > 0$, i.e. A_i has occurred at least once, then A_i is called a *focal element*.

According to [15], the *belief* $Bel(A)$ and *plausibility* $Pl(A)$ measures of an event $A \subseteq U$ can be defined as

$$\mathrm{Bel}(A) = \sum_{A_i : A_i \subseteq A} m(A_i), \quad \mathrm{Pl}(A) = \sum_{A_i : A_i \cap A \neq \varnothing} m(A_i). \tag{3}$$

As pointed out in [7], a belief function can formally be defined as a function satisfying axioms which can be viewed as a weakening of the Kolmogorov axioms that characterize probability functions. Therefore, it seems reasonable to understand a belief function as a generalized probability function [6] and the belief $Bel(A)$ and plausibility $Pl(A)$ measures can be regarded as lower and upper bounds for the probability of A, i.e., $\mathrm{Bel}(A) \leq \Pr(A) \leq \mathrm{Pl}(A)$.

If there are r independent different sources of evidence, then Dempster's rule of combination of evidence can be used for computing combined BPA's. Dempster's rule combines multiple belief functions through their BPA's. Let $m_q(A_i^{(q)})$, $i = 1, ..., n_q$, be the BPAs of n_q focal elements $A_1^{(q)}, ..., A_{n_q}^{(q)}$ obtained from the q-th source of evidence. Then a combined BPA, $m(B)$, of a set B is given by

$$m(B) = \frac{1}{1-K} \sum_{A_i^{(1)} \cap ... \cap A_j^{(q)} = B} \prod_{q=1}^{r} m_q(A_i^{(q)}), \tag{4}$$

where

$$K = \sum_{A_i^{(1)} \cap ... \cap A_j^{(q)} = \varnothing} \prod_{q=1}^{r} m_q(A_i^{(q)}). \tag{5}$$

K represents basic probability mass associated with conflict. Note that Dempster's rule can not be used in case of $K = 1$, i.e., conflicting evidence can not be combined.

3 The ranking procedure

We again suppose that there is a set of alternatives $\mathbb{A} = \{A_1, ..., A_n\}$ consisting of n elements. An expert chooses some subset $B_k \in \mathcal{P}o(\mathbb{A})$ of alternatives from the power set $\mathcal{P}o(\mathbb{A})$ and compares this subset with another subset $B_i \in \mathcal{P}o(\mathbb{A})$ of alternatives. In other words, experts set up the preferences $B_k \succeq B_i$ or $B_i \succeq B_k$. Every expert choice adds "1" to the corresponding preference. Methods of giving rewards 1 when predictions are "correct" and discounting the predictions when they are "wrong" are similar to the so-called re-enforcement learning [12].

For example, if $\mathbb{A} = \{A_1, A_2, A_3\}$, $B_k = \{A_3\}$, and $B_i = \{A_1, A_3\}$, then the preference $B_k \succeq B_i$ means that the expert chooses the alternative A_3 from alternatives A_1 and A_3. This is equivalent to the preference $\{A_3\} \succeq \{A_1\}$. If $B_k = \{A_2\}$ and $B_i = \{A_1, A_3\}$, then the preference $B_k \succeq B_i$ means that the alternative A_2 is more preferable than A_1 or A_3. In such the way, the experts compare some groups of alternatives from the set $\mathbb{A}$.

The extended matrix of pairwise comparisons in this case has $2^n - 1$ columns and $2^n - 1$ rows (the empty element of $\mathcal{P}o(\mathbb{A})$ is not considered here). An example of such the matrix by $n = 2$ is shown in Table 1. It is supposed that experts only compare subsets of alternatives, but they do not provide preference values or weights of preferences. At that, if an expert supplies the comparison assessment $B_k \succeq B_i$, then the value 1 is added to the corresponding cell in the comparison matrix (k-th row and i-th column). In this case, the preference values c_{ki} can be regarded as the number of experts chosen the comparison assessment $B_k \succeq B_i$.

The following question is how to process the obtained matrix of pairwise comparisons.

First, we define the set $\mathcal{L}$ of *basic preferences*

$$\mathcal{L} = \{\{A_i\} \succeq \{A_k\},\ \forall i, k \in \{1, 2, ..., n\},\ i \neq k\}.$$

We also define the set of all subsets of preferences $\mathcal{M}$, which consists of preferences of the form $B_k \succeq B_i$.

Table 1: The extended matrix of pairwise comparisons

	$\{A_1\}$	$\{A_2\}$	$\{A_1, A_2\}$
$\{A_1\}$	-	c_{12}	c_{13}
$\{A_2\}$	c_{21}	-	c_{23}
$\{A_1, A_2\}$	c_{31}	c_{32}	-

Note that the preferences $\{A_i\} \succeq \{A_k\}$ and $\{A_j\} \succeq \{A_k\}$ follow from the preference $\{A_i, A_j\} \succeq \{A_k\}$ if $i \neq k$ and $j \neq k$. By generalizing the above and assuming that $B_k = \{A_v, ..., A_w\}$ consists of n_k alternatives and $B_i = \{A_t, ..., A_l\}$ consists of n_i alternatives such that $B_k \cap B_i = \varnothing$, we can say that the preference $B_k \succeq B_i$ implies $n_k \cdot n_i$ basic preferences of the form

$$A_v \succeq A_t, A_{v+1} \succeq A_t, ..., A_v \succeq A_l, ..., A_w \succeq A_t, ..., A_w \succeq A_l.$$

At the same time, the basic preference $\{A_j\} \succeq \{A_k\}$ follows from the preference $\{A_k, A_j\} \succeq \{A_k\}$ if $j \neq k$. It can be seen from the above that the comparison of common parts of subsets B_k and B_i makes no sense. Experts have to compare different subsets of alternatives, i.e., $B_k \cap B_i = \varnothing$ for all k and i. Nevertheless, we would not like to restrict experts to supply only "permitted" judgments. However, by processing the preference $B_k \succeq B_i$ with $B_k \cap B_i = \widetilde{B}_{ki} \neq \varnothing$, we will replace them by the following two preferences $B_k \succeq B_i \backslash \widetilde{B}_{ki}$ and $B_k \backslash \widetilde{B}_{ki} \succeq B_i$. For instance, the preference $\{A_1, A_2\} \succeq \{A_1, A_2, A_3\}$ can be represented as the preference $\{A_1, A_2\} \succeq \{A_3\}$ ($B_k \succeq B_i \backslash \widetilde{B}_{ki}$, $\widetilde{B}_{ki} = \{A_1 A_2\}$), which can be represented as the subset of the set $\mathcal{L}$ consisting of the basic preferences $\{A_1\} \succeq \{A_3\}$, $\{A_2\} \succeq \{A_3\}$. Therefore, throughout the paper we will assume that experts supply only "permitted" preferences.

In sum, we can represent every preference by the set of simplest preferences of the form $\{A_i\} \succeq \{A_k\}$ from the set $\mathcal{L}$. For brevity, we will denote the preference $B_k \succeq B_i$ by $\mathcal{B}_{ki}$.

If we restrict the extended comparison matrix only by basic preferences from the set $\mathcal{L}$, then, by using the standard multinomial model, we can assign some probability to every cell of the matrix (to every pairwise comparison). Then the set $\mathcal{L}$ can be regarded as the universal set, which with other preferences can be regarded as the "power" set[2] $\mathcal{M}$. At the same time, we can not precisely assign probabilities to elements of the universal set by using only the restricted comparison matrix because it is necessary to take into account the estimates concerning the groups of alternatives. We do not have complete probabilistic information about basic preferences from the set $\mathcal{L}$, and we do not know how the assessments concerning the groups of alternatives impact on the probabilities of elements from $\mathcal{L}$. For instance, by having the probability, say p, for the preference $\{A_2\} \succeq \{A_1 A_3\}$ and by representing this preference as a subset of two basic preferences $\{A_2\} \succeq \{A_1\}$ and $\{A_2\} \succeq \{A_3\}$, we do not know how the probability p is distributed among the preferences $\{A_2\} \succeq \{A_1\}$ and $\{A_2\} \succeq \{A_3\}$. The same situation takes place when we analyze evidence in the framework of DST. Consequently, we can apply DST to the considered sets of preferences.

[2] As a matter of fact, the set $\mathcal{M}$ is not the power set because it consists only of a part of all subsets produced by comparisons.

For every pairwise comparison in the extended comparison matrix, we define its BPA as follows:

$$m(B_k \succeq B_i) = m(\mathcal{B}_{ki}) = \frac{c_{ki}}{N},\ N = \sum_{k,i\in\{1,2,...,n\},\ k\neq i} c_{ki}.$$

Since every preference $B_k \succeq B_i$ is represented as a set of basic preferences, we denote this set of basic preferences $\mathcal{L}_{ki} \subseteq \mathcal{L}$. Now we can define some rules of set-theoretic operations.

We will say that the preference $B_j \succeq B_l$ is a subset of the preference $B_k \succeq B_i$ or $B_k \succeq B_i$ includes $B_j \succeq B_l$ if there holds $\mathcal{L}_{jl} \subseteq \mathcal{L}_{ki}$. This means that the set of basic preferences $\mathcal{L}_{jl}$ produced by $B_j \succeq B_l$ is a subset of basic preferences $\mathcal{L}_{ki}$ produced by $B_k \succeq B_i$.

We will also say that the preference $B_j \succeq B_l$ intersects the preference $B_k \succeq B_i$ if there holds $\mathcal{L}_{ki} \cap \mathcal{L}_{jl} \neq \varnothing$. This means that the sets $\mathcal{L}_{jl}$ and $\mathcal{L}_{ki}$ produced by $B_j \succeq B_l$ and $B_k \succeq B_i$, respectively, have common basic preferences.

Then the belief and plausibility functions can be defined for the preference $B_k \succeq B_i$ as follows:

$$\mathrm{Bel}(\mathcal{B}_{ki}) = \sum_{j,l:\mathcal{L}_{jl}\subseteq\mathcal{L}_{ki}} m(\mathcal{B}_{jl}),$$

$$\mathrm{Pl}(\mathcal{B}_{ki}) = \sum_{j,l:\mathcal{L}_{jl}\cap\mathcal{L}_{ki}\neq\varnothing} m(\mathcal{B}_{jl}).$$

Every expert is not regarded as an independent source of information here because expert judgments may be conflicting. This fact does not allow us to use Dempster rule for combining the sources. Nevertheless, we can consider groups of experts as the independent sources of information. The case of independent groups of experts is studied in Section 4.

The next question is how to rank alternatives by using the proposed framework. The belief and plausibility functions of the preferences $B_k \succeq B_i$ do not give a possibility to directly choose an optimal alternative from the set of alternatives $\mathbb{A}$. Therefore, two approaches can be proposed for ranking the alternatives.

3.1 Comparison with the whole set of alternatives

One of the possible solutions is to consider preferences of the form $B_k \succeq \mathbb{A}$. A nice idea of Beynon *et al* [3, 4] in the DS/AHP method is that the comparison of groups of alternatives with a whole set of alternatives is equivalent to the identification of the most favorable alternatives from the set $\mathbb{A}$. This type of preferences means that an expert prefers the subset B_k to all possible alternatives $\mathbb{A}$. For instance, if $\mathbb{A} = \{A_1, A_2, A_3\}$, then the preference $\{A_1\} \succeq \{A_1A_2A_3\}$ means that an expert chooses A_1 from all alternatives $\{A_1A_2A_3\}$. This implies that, by taking $B_k = \{A_k\}$ and by computing the belief and plausibility functions of the preference $\{A_k\} \succeq \mathbb{A}$, we determine the lower and upper probabilities of the alternative A_k. Consequently, by computing belief and plausibility functions for all preferences $\{A_k\} \succeq \mathbb{A}$, $k = 1, ..., n$, we can rank all alternatives.

Example 1 *Suppose that there are three types of transport (three alternatives $\mathbb{A} = \{A_1, A_2, A_3\}$): rail transport A_1, motor transport A_2, water transport A_3. Experts provide the following preferences:*

Table 2: The correspondences between subsets of alternatives and their short notations

$\{A_1\}$	$\{A_2\}$	$\{A_3\}$	$\{A_1, A_2\}$	$\{A_1, A_3\}$	$\{A_2, A_3\}$	$\{A_1, A_2, A_3\}$
B_1	B_2	B_3	B_4	B_5	B_6	B_7

five experts ($c_1 = 5$)*:* $\{A_1, A_3\} \succeq \{A_2\} = \mathcal{B}_{52}$,
two experts ($c_2 = 2$)*:* $\{A_3\} \succeq \{A_1, A_2, A_3\} = \mathcal{B}_{37}$,
three experts ($c_3 = 3$)*:* $\{A_1\} \succeq \{A_3\} = \mathcal{B}_{13}$.

The correspondences between subsets of alternatives and their short notations B_k *are given in Table 2.*

The total number of estimates is $N = 10$. *This implies that the BPA's of focal elements are* $m(\mathcal{B}_{52}) = 0.5$, $m(\mathcal{B}_{37}) = 0.2$, $m(\mathcal{B}_{13}) = 0.3$. *Note that the preferences can be represented by the sets*

$$\mathcal{L}_{52} = \{\{A_1\} \succeq \{A_2\}, \{A_3\} \succeq \{A_2\}\},$$
$$\mathcal{L}_{37} = \{\{A_3\} \succeq \{A_1\}, \{A_3\} \succeq \{A_2\}\},$$
$$\mathcal{L}_{13} = \{\{A_1\} \succeq \{A_3\}\}.$$

Hence the belief functions of the preferences $\{A_k\} \succeq \mathbb{A}$, $\mathcal{L}_{k7} = \{\{A_k\} \succeq \{A_i\},\ i = 1, 2, 3\}$, $k = 1, 2, 3$, *are*

$$\mathrm{Bel}(\{A_1\} \succeq \mathbb{A}) = m(\mathcal{B}_{13}) = 0.3,$$
$$\mathrm{Bel}(\{A_2\} \succeq \mathbb{A}) = 0,$$
$$\mathrm{Bel}(\{A_3\} \succeq \mathbb{A}) = m(\mathcal{B}_{37}) = 0.2.$$

The plausibility functions of the same preferences are

$$\mathrm{Pl}(\{A_1\} \succeq \mathbb{A}) = m(\mathcal{B}_{52}) + m(\mathcal{B}_{13}) = 0.8,$$
$$\mathrm{Pl}(\{A_2\} \succeq \mathbb{A}) = 0,$$
$$\mathrm{Pl}(\{A_3\} \succeq \mathbb{A}) = m(\mathcal{B}_{52}) + m(\mathcal{B}_{37}) = 0.7.$$

It can be seen from the obtained numerical results that the first alternative is the most preferable. Moreover, the "best" ranking of the alternatives is

$$A_1 \succeq A_3 \succeq A_2.$$

It should be noted that the choice of the "best" ranking in the above numerical example does not meet difficulties because we can explicitly sort intervals of the belief and plausibility functions. However, there are cases when this problem can not be solved in the same simple way. Therefore, we have to define a rule for sorting the alternatives. In other words, we have to find a rule in order to compare overlapping intervals. There exist a lot of methods for comparison of the overlapping intervals. Nevertheless, we consider one of the most attractive and justified methods using the so-called caution parameter [14, 22] or the parameter of pessimism η which has the same meaning as the optimism parameter in Hurwicz criterion [10]. According to this method, the "best" ranking from all possible ones should be chosen in such a way that makes the convex combination $\eta \cdot Bel(B) + (1-\eta)Pl(B)$ achieve its maximum. Here $\eta \in [0,1]$ is the caution parameter. If $\eta = 1$, then we analyze only belief functions and make pessimistic decision. This type of decision is very often used [1, 11]. If $\eta = 0$, then we analyze only plausibility functions and make optimistic decision.

3.2 Belief and plausibility functions of different rankings

The second approach for ranking the alternatives is based on computing the belief and plausibility functions of subsets of $\mathbb{A}$ in accordance with all possible sequences of indices $(i_1, ..., i_n)$ corresponding to (1). In other words, we have to find belief and plausibility functions of subsets produced by preferences

$$B(i_1, ..., i_n) = \{A_{i_1} \succeq A_{i_2},\ A_{i_2} \succeq A_{i_3}, ..., A_{i_{n-1}} \succeq A_{i_n}\}$$

for all possible $(i_1, ..., i_n)$. After the above functions are computed, the optimal ranking is chosen in accordance with some rule of comparison of intervals $(\mathrm{Bel}, \mathrm{Pl})$.

Note that

$$B(i_1, ..., i_n) = \mathcal{B}_{i_1,i_2} \cap \mathcal{B}_{i_2,i_3} \cap ... \cap \mathcal{B}_{i_{n-1},i_n}.$$

This implies that it is impossible to find a set of preferences $\mathcal{L}_{jl}$ among the "permitted" preferences, which would be included in $B(i_1, ..., i_n)$. Consequently, there holds $Bel(B) = 0$.

On the other hand, the set B intersects the following preferences:

$$\begin{aligned} &A_{i_1} \succeq A_{i_2}, A_{i_1} \succeq A_{i_3}, ..., A_{i_1} \succeq A_{i_n}, \\ &A_{i_2} \succeq A_{i_3}, ..., A_{i_2} \succeq A_{i_n}, \\ &\quad ... \\ &A_{i_{n-1}} \succeq A_{i_n}. \end{aligned}$$

Denote the above set of preferences by $\mathcal{C}(i_1, ..., i_n)$. Then

$$\mathrm{Pl}(B(i_1, ..., i_n)) = \sum_{j,l:\mathcal{L}_{jl} \cap \mathcal{C}(i_1,...,i_n) \neq \varnothing} m(\mathcal{B}_{jl}).$$

Since the belief function is 0 for arbitrary sets of indices $(i_1, ..., i_n)$, we compare different rankings $(i_1, ..., i_n)$ only by comparing the plausibility functions. The rule for determining the "best" ranking is the following. The "best" ranking from all possible ones should be chosen in such a way that makes $\mathrm{Pl}(B(i_1, ..., i_n))$ achieve its maximum.

Example 2 *Let us return to Example 1 and rank three alternatives by using the plausibility functions of $B(i_1, ..., i_n)$. We have 6 possible sequences of indices (i_1, i_2, i_3). The plausibility* *functions are*

$$\begin{aligned}
\text{Pl}(B(1,2,3)) &= m(\mathcal{B}_{52}) + m(\mathcal{B}_{13}) = 0.8, \\
\text{Pl}(B(1,3,2)) &= m(\mathcal{B}_{52}) + m(\mathcal{B}_{37}) + m(\mathcal{B}_{13}) = 1, \\
\text{Pl}(B(2,1,3)) &= m(\mathcal{B}_{13}) = 0.3, \\
\text{Pl}(B(2,3,1)) &= m(\mathcal{B}_{37}) = 0.2, \\
\text{Pl}(B(3,1,2)) &= m(\mathcal{B}_{52}) + m(\mathcal{B}_{37}) = 0.7, \\
\text{Pl}(B(3,2,1)) &= m(\mathcal{B}_{52}) + m(\mathcal{B}_{37}) = 0.7.
\end{aligned}$$

It can be seen from the obtained numerical results that the "best" ranking of the alternatives has the unit plausibility function and is

$$A_1 \succeq A_3 \succeq A_2.$$

4 Independent groups of experts

If there are several independent groups of experts, then the pairwise comparisons elicited in different groups have to be combined. We consider every group as a source of evidence. If all sources are absolutely reliable, then one of the most well-established and frequently used methods for combining the independent sources of evidence is Dempster's rule of combination.

Without loss of generality, we assume that there are two sources of evidence. Let us define Dempster's rule of combination in the framework of the considered sets of preferences. Denote the preferences obtained from the first and the second sources by upper indices (1) and (2), respectively. The combined BPA of the preference $B_k \succeq B_i$ is determined as

$$m_{12}(\mathcal{B}_{ki}) = \frac{1}{1-K} \sum_{j,l,v,w:\mathcal{L}_{jl}^{(1)} \cap \mathcal{L}_{vw}^{(2)} = \mathcal{L}_{ki}} m_1(\mathcal{B}_{jl}^{(1)}) \cdot m_2(\mathcal{B}_{vw}^{(2)}),$$

where the weight of conflict is

$$K = \sum_{\mathcal{L}_{jl}^{(1)} \cap \mathcal{L}_{vw}^{(2)} = \varnothing} m_1(\mathcal{B}_{jl}^{(1)}) \cdot m_2(\mathcal{B}_{vw}^{(2)}).$$

Example 3 *Let us return to Example 1. Suppose that two independent groups of experts provide the following comparative judgments.*

The first group: $\{A_1\} \succeq \{A_2A_3\} = \mathcal{B}_{16}^{(1)}$, $c_{11} = 2$, $\{A_1A_2\} \succeq \{A_1A_2A_3\} = \mathcal{B}_{47}^{(1)}$, $c_{12} = 3$.

The second group[3]*:* $\{A_1A_3\} \succeq \{A_2\} = \mathcal{B}_{52}^{(2)}$, $c_{21} = 5$, $\{A_3\} \succeq \{A_1A_2A_3\} = \mathcal{B}_{37}^{(2)}$, $c_{22} = 2$, $\{A_1\} \succeq \{A_3\} = \mathcal{B}_{13}^{(2)}$, $c_{23} = 3$.

Note that the condition $\{A_1A_2\} \succeq \{A_1A_2A_3\}$ *is equivalent to the "permitted" preference* $\{A_1A_2\} \succeq \{A_3\}$, *and the preference* $\{A_3\} \succeq \{A_1A_2A_3\}$ *is equivalent to the "permitted" preference* $\{A_3\} \succeq \{A_1A_2\}$, *i.e.,* $\mathcal{B}_{47}^{(1)} = \mathcal{B}_{43}^{(1)}$ *and* $\mathcal{B}_{37}^{(2)} = \mathcal{B}_{34}^{(2)}$. *The BPA's of all preferences are:*

$$m_1(\mathcal{B}_{16}^{(1)}) = 0.4,\ m_1(\mathcal{B}_{43}^{(1)}) = 0.6,$$

[3] The second group has been considered in Examples 1 and 2.

Table 3: The preference intersections for Dempster's combination rule

			The first group	
			$\{A_1\} \succeq \{A_2A_3\}$	$\{A_1A_2\} \succeq \{A_3\}$
			$\mathcal{B}_{16}^{(1)}$	$\mathcal{B}_{43}^{(1)}$
The second group	$\{A_1A_3\} \succeq \{A_2\}$	$\mathcal{B}_{52}^{(2)}$	$\{A_1\} \succeq \{A_2\}$	$\varnothing$
	$\{A_3\} \succeq \{A_1A_2\}$	$\mathcal{B}_{34}^{(2)}$	$\varnothing$	$\varnothing$
	$\{A_1\} \succeq \{A_3\}$	$\mathcal{B}_{13}^{(2)}$	$\{A_1\} \succeq \{A_3\}$	$\{A_1\} \succeq \{A_3\}$

$$m_2(\mathcal{B}_{52}^{(2)}) = 0.5,\ m_2(\mathcal{B}_{34}^{(2)}) = 0.2,\ m_2(\mathcal{B}_{13}^{(2)}) = 0.3.$$

Now we define the sets $\mathcal{L}_{ki}^{(m)}$ *for the first (*$m = 1$*) and second (*$m = 2$*) groups.*

The first group: $\mathcal{L}_{16}^{(1)} = \{\{A_1\} \succeq \{A_2\}, \{A_1\} \succeq \{A_3\}\}$, $\mathcal{L}_{43}^{(1)} = \{\{A_1\} \succeq \{A_3\}, \{A_2\} \succeq \{A_3\}\}$.

The second group: $\mathcal{L}_{52}^{(2)} = \{\{A_1\} \succeq \{A_2\}, \{A_3\} \succeq \{A_2\}\}$, $\mathcal{L}_{34}^{(2)} = \{\{A_3\} \succeq \{A_1\}, \{A_3\} \succeq \{A_2\}\}$, $\mathcal{L}_{13}^{(2)} = \{\{A_1\} \succeq \{A_3\}\}$.

Possible intersections for realizing Dempster's combination rule are shown in Table 3. Hence, we compute the weight of conflict

$$\begin{aligned} K &= m_1(\mathcal{B}_{16}^{(1)}) \cdot m_2(\mathcal{B}_{34}^{(2)}) + m_1(\mathcal{B}_{43}^{(1)}) \cdot m_2(\mathcal{B}_{52}^{(2)}) + m_1(\mathcal{B}_{43}^{(1)}) \cdot m_2(\mathcal{B}_{34}^{(2)}) \\ &= 0.4 \cdot 0.2 + 0.6 \cdot 0.5 + 0.6 \cdot 0.2 = 0.5. \end{aligned}$$

The large value of the weight of conflict reflects the fact that the expert judgments from different groups are contradictory. In sum, we have only two non-zero combined BPA's of the preferences $\mathcal{B}_{12}^{(12)} = \{A_1\} \succeq \{A_2\}$ *and* $\mathcal{B}_{13}^{(12)} = \{A_1\} \succeq \{A_3\}$*, which are computed as follows:*

$$\begin{aligned} m_{12}(\mathcal{B}_{12}^{(12)}) &= \frac{1}{1-K} \cdot m_1(\mathcal{B}_{16}^{(1)}) \cdot m_2(\mathcal{B}_{52}^{(2)}) \\ &= 2 \cdot 0.4 \cdot 0.5 = 0.4, \end{aligned}$$

$$\begin{aligned} m_{12}(\mathcal{B}_{13}^{(12)}) &= \frac{1}{1-K} \cdot \left(m_1(\mathcal{B}_{16}^{(1)}) + m_1(\mathcal{B}_{43}^{(1)}) \right) \cdot m_2(\mathcal{B}_{13}^{(2)}) \\ &= 2 \cdot 0.3 = 0.6. \end{aligned}$$

Let us compute now the belief and plausibility functions of alternatives A_1, A_2, A_3 *or preferences* $\{A_i\} \succeq \mathbb{A}$, $i = 1, 2, 3$*. The belief and plausibility functions of* $\{A_1\} \succeq \mathbb{A}$ *are*

$$\mathrm{Bel}(\{A_1\} \succeq \mathbb{A}) = m_{12}(\mathcal{B}_{12}^{(12)}) + m_{12}(\mathcal{B}_{13}^{(12)}) = 1,$$

$$\mathrm{Pl}(\{A_1\} \succeq \mathbb{A}) = 1.$$

The belief and plausibility functions of $\{A_2\} \succeq \mathbb{A}$ *and* $\{A_3\} \succeq \mathbb{A}$ *are* 0*. It follows from the obtained results that the "best" alternative is* A_1*. But we can not completely rank all the alternatives.*

One of the main reasons of the situation in Example 3 when it is impossible to find a unique "best" ranking is the small number of expert judgments. Another reason is the used assumption that the sources of evidence are absolutely reliable. If we know that the sources are unreliable, then modified or discounted Dempster's rule of combination can be used. It should be noted that discounted Dempster's combination rule was proposed by Shafer [15] for avoiding the conflicting situations, for taking into account the reliability of sources of evidence and achieving $K \neq 1$. Shafer [15] proposed to use discounting the BPA's of the q-th source of evidence with the discount rate $\alpha_q \in [0,1]$ characterizing the reliability of the source. The discounting is carried out by multiplying the BPA's of all focal elements of the q-th source by α_q. As a result, we obtain a new BPA's $m_q^{\alpha}(A) = \alpha_q \cdot m_q(A)$ for every focal element A. If the discount rate is 0, then the corresponding source of evidence is absolutely unreliable and $m_q^{\alpha}(A) = 0$. If $\alpha_q = 1$, then the source is absolutely reliable and $m_q^{\alpha}(A) = m_q(A)$.

The condition of the unit sum of BPA's implies that the additional BPA of $\mathbb{A}$ is $m_q^{\alpha}(\mathbb{A}) = (1 - \alpha_q) + \alpha_q \cdot m_q(\mathbb{A})$. In fact, the non-zero additional BPA of $\mathbb{A}$ does not change the available information.

The next questions are how to determine the discount rate for a source of evidence and how to take into account the possible fact that the number of expert judgments is very small. One of the ways for asking the above questions is to use the so-called imprecise Dirichlet model and getting the cautious rankings.

5 Cautious ranking with the imprecise Dirichlet model

One of the main difficulty of the proposed ranking method is the possible small number of experts. Expression (2) can be used when the number of expert judgments is rather large. When we have a small number of judgments, (2) might give incorrect BPA's. In order to overcome this difficulty, the *imprecise Dirichlet model* (IDM) [21] can be applied to extend belief and plausibility functions such that a lack of sufficient statistical data can be taken into account [17, 18].

For brevity, we will not consider in detail what this model is and how to obtain it. The interested reader should refer to [2, 21] and [17, 18, 19, 20]. We point out only that the use of the IDM leads to the extended belief and plausibility functions of the form:

$$\mathrm{Bel}_s(A) = \frac{N \cdot \mathrm{Bel}(A)}{N + s},\ \mathrm{Pl}_s(A) = \frac{N \cdot \mathrm{Pl}(A) + s}{N + s}.$$

Here the *hyperparameter* s determines how quickly upper and lower probabilities of events converge as statistical data accumulate; N is the number of expert judgments. Smaller values of s produce faster convergence and stronger conclusions, whereas large values of s produce more cautious inferences. Walley [21] and Bernard [2] argue that the parameter s should be taken to be 1 or 2.

It should be noted that the simple modification of the belief and plausibility functions with using the IDM has a number of nice properties [18]. For example, if we have N identical estimates, then the belief and plausibility functions are the same $Bel(A) = \mathrm{Pl}(A) = 1$. This implies that the belief and plausibility functions do not depend on the value N while $Bel_s(A)$ and $\mathrm{Pl}_s(A)$ are $N/(N+s)$ and 1, respectively.

However, the main advantage of the IDM is that it produces the cautious inference. In particular, if $N=0$, then $Bel_s(A)=0$ and $\mathrm{Pl}_s(A)=1$. In the case $N\to\infty$, it can be stated for any s: $Bel_s(A) = Bel(A)$, $\mathrm{Pl}_s(A)=\mathrm{Pl}(A)$.

The extended belief and plausibility functions are obtained from the BPA's $m^*(A)=c/(N+s)$ for every A and the additional BPA $m^*(\mathbb{A}\succeq\mathbb{A})=s/(N+s)$, i.e., $Bel_s(A)$ and $\mathrm{Pl}_s(A)$ can be obtained as standard belief and plausibility functions under condition that there are s additional observations $A=\mathbb{A}\succeq\mathbb{A}$. This fact allows us to change the BPA's of B_i and to make the cautious decision.

Denote $\varkappa = N/(N+s)$. Then there holds $m^*(A)=\varkappa\cdot m(A_i)$. One can see from the last expression for $m^*(A)$ that $\varkappa$ is the discount rate characterizing the reliability of a source of evidence. Consequently, the discount rate depends on the number of estimates N. If $N=0$, then $\varkappa=0$ and the corresponding source of evidence is absolutely unreliable. If $N\to\infty$, then $\varkappa=1$ and the source is absolutely reliable. For instance, the first and second sources of evidence in Example 3 by $s=1$ have the discount rates $\alpha_1=5/6\simeq 0.83$ and $\alpha_2=10/11\simeq 0.91$, respectively.

Example 4 *Let us return to Example 3 by taking into account the reliability of sources of evidence and by using the IDM. The modified BPA's by $s=1$ are*

$$m_1^*(\mathcal{B}_{16}^{(1)})=0.33,\ m_1^*(\mathcal{B}_{43}^{(1)})=0.5,\ m_1^*(\mathcal{B}_{77}^{(1)})=0.17,$$

$$m_2^*(\mathcal{B}_{52}^{(2)})=0.46,\ m_2^*(\mathcal{B}_{34}^{(2)})=0.18,\ m_2^*(\mathcal{B}_{13}^{(2)})=0.27,\ m_2^*(\mathcal{B}_{77}^{(2)})=0.09.$$

Here $\mathcal{B}_{77}^{(1)}=\mathcal{B}_{77}^{(2)}=\mathbb{A}\succeq\mathbb{A}$. According to [18], the modified weight of conflict

$$K^*=\alpha_1\alpha_2K=0.83\cdot 0.91\cdot 0.5=0.378.$$

Possible intersections for realizing Dempster's combination rule are shown in Table 4. The combined BPA's of $\mathcal{B}_{12}^{(12)}$ and $\mathcal{B}_{13}^{(12)}$ are

$$\begin{aligned}m_{12}(\mathcal{B}_{12}^{(12)}) &= \frac{1}{1-K^*}\cdot m_1^*(\mathcal{B}_{16}^{(1)})\cdot m_2^*(\mathcal{B}_{52}^{(2)})\\ &= 1.61\cdot 0.33\cdot 0.46=0.244,\end{aligned}$$

$$\begin{aligned}m_{12}(\mathcal{B}_{13}^{(12)}) &= \frac{1}{1-K^*}\cdot\Big(m_1^*(\mathcal{B}_{16}^{(1)})+m_1^*(\mathcal{B}_{43}^{(1)})+m_1^*(\mathcal{B}_{77}^{(1)})\Big)\cdot m_2^*(\mathcal{B}_{13}^{(2)})\\ &= 1.61\cdot 0.27=0.435.\end{aligned}$$

The combined BPA's of other preferences from the Table 4 are similarly computed

$$m_{12}(\mathcal{B}_{16}^{(12)})=0.048,\ m_{12}(\mathcal{B}_{43}^{(12)})=0.072,$$

$$m_{12}(\mathcal{B}_{52}^{(12)})=0.126,\ m_{12}(\mathcal{B}_{34}^{(12)})=0.049,\ m_{12}(\mathcal{B}_{77}^{(12)})=0.026.$$

The belief and plausibility functions of $\{A_k\}\succeq\mathbb{A}$ are

$$\mathrm{Bel}_1(\{A_1\}\succeq\mathbb{A})=m_{12}(\mathcal{B}_{12}^{(12)})+m_{12}(\mathcal{B}_{13}^{(12)})+m(\mathcal{B}_{16}^{(12)})=0.727,$$

Table 4: The preference intersections for modified Dempster's combination rule

		The first group		
		$\mathcal{B}_{16}^{(1)}$	$\mathcal{B}_{43}^{(1)}$	$\mathcal{B}_{77}^{(1)}$
The	$\mathcal{B}_{52}^{(2)}$	$\{A_1\} \succeq \{A_2\}$	$\varnothing$	$\{A_1A_3\} \succeq \{A_2\}$
second	$\mathcal{B}_{34}^{(2)}$	$\varnothing$	$\varnothing$	$\{A_3\} \succeq \{A_1A_2\}$
group	$\mathcal{B}_{13}^{(2)}$	$\{A_1\} \succeq \{A_3\}$	$\{A_1\} \succeq \{A_3\}$	$\{A_1\} \succeq \{A_3\}$
	$\mathcal{B}_{77}^{(2)}$	$\{A_1\} \succeq \{A_2A_3\}$	$\{A_1A_2\} \succeq \{A_3\}$	$\mathbb{A} \succeq \mathbb{A}$

$$\mathrm{Pl}_1(\{A_1\} \succeq \mathbb{A}) = \mathrm{Bel}_1(\{A_1\} \succeq \mathbb{A}) + m_{12}(\mathcal{B}_{52}^{(12)}) + m_{12}(\mathcal{B}_{43}^{(12)}) + m_{12}(\mathcal{B}_{77}^{(12)}) = 0.951,$$

$$\mathrm{Bel}_1(\{A_2\} \succeq \mathbb{A}) = 0,\ \mathrm{Pl}_1(\{A_2\} \succeq \mathbb{A}) = m_{12}(\mathcal{B}_{43}^{(12)}) + m_{12}(\mathcal{B}_{77}^{(12)}) = 0.098,$$

$$\mathrm{Bel}_1(\{A_3\} \succeq \mathbb{A}) = m_{12}(\mathcal{B}_{34}^{(12)}) = 0.049,$$

$$\mathrm{Pl}_1(\{A_3\} \succeq \mathbb{A}) = m_{12}(\mathcal{B}_{34}^{(12)}) + m_{12}(\mathcal{B}_{52}^{(12)}) + m_{12}(\mathcal{B}_{77}^{(12)}) = 0.201.$$

It can be seen from the above results that the "best" ranking is

$$A_1 \succeq A_3 \succeq A_2.$$

6 Conclusion

The method for ranking of alternatives or objects has been proposed in the paper. The main feature of the method is that it allows us to deal with comparisons of arbitrary groups of alternatives. Some algebra of preferences with the set of necessary set-theoretical operations has been introduced. This algebra gives the possibility to use the framework DST and to compute the belief and plausibility functions of alternatives or rankings. Moreover, the proposed method has been extended on the case of independent groups of experts which have been combined by means of Dempster's combination rule. In order to make cautious decisions when the number of expert estimates is rather small and to take into account objectively the reliability of different groups of experts, the IDM is used. The numerical examples illustrate the method and its extensions.

At the same time, there are open questions. The first question is how to determine the most "reasonable" value of the hyperparameter s. This is a direction for further work. Another direction for further work is how to take into account the possible rates of comparisons, which experts could provide in some cases. It should be noted that there are many combination rules in the DST in addition to Dempster's combination rule. The choice of the best rule taking into account the possible dependence of groups of experts is also a question for further research.

Acknowledgement

I would like to express my appreciation to the anonymous referees whose very valuable comments have improved the paper.

References

[1] T. Augustin. Expected utility within a generalized concept of probability - a comprehensive framework for decision making under ambiguity. *Statistical Papers*, 43:5–22, 2002.

[2] J.-M. Bernard. An introduction to the imprecise Dirichlet model for multinomial data. *International Journal of Approximate Reasoning*, 39(2-5):123–150, 2005.

[3] M. Beynon. DS/AHP method: A mathematical analysis, including an understanding of uncertainty. *European Journal of Operational Research*, 140:148–164, 2002.

[4] M. Beynon, B. Curry, and P. Morgan. The Dempster-Shafer theory of evidence: An alternative approach to multicriteria decision modelling. *Omega*, 28:37–50, 2000.

[5] R.M. Cook. *Experts in Uncertainty. Opinion and Subjective Probability in Science*. Oxford University Press, New York, 1991.

[6] A.P. Dempster. Upper and lower probabilities induced by a multi-valued mapping. *Annales of Mathematical Statistics*, 38:325–339, 1967.

[7] J.Y. Halpern and R. Fagin. Two views of belief: Belief as generalized probability and belief as evidence. *Artificial Intelligence*, 54:275–317, 1992.

[8] E. Hullermeier and J. Furnkranz. Comparison of ranking procedures in pairwise preference learning. In *Proc. of the 10th International Conference on Information Processing and Management of Uncertainty in Knoweldge-Based Systems, IPMU-04*, Perugua, Italy, 2004.

[9] E. Hullermeier and J. Furnkranz. Ranking by pairwise comparison: A note on risk minimization. In *Proc. of the IEEE International Conference on Fuzzy Systems, FUZZ-IEEE-04*, Budapest, Hungary, July 2004.

[10] R.D. Luce and H. Raiffa. *Games and decisions*. Wiley, New York, 1957.

[11] K.-M. Osei-Bryson. Supporting knowledge elicitation and consensus building for Dempster-Shafer decision models. *International Journal of Intelligent Systems*, 18:129–148, 2003.

[12] S. Russell and P. Norvig. *Artificial Intelligence: A Modern Approach*. Prentice Hall, 2002.

[13] T.L. Saaty. *Multicriteria Decision Making: The Analytic Hierarchy Process*. McGraw Hill, New York, 1980.

[14] J. Schubert. On ρ in a decision-theoretic apparatus of Dempster-Shafer theory. *International Journal of Approximate Reasoning*, 13:185–200, 1995.

[15] G. Shafer. *A Mathematical Theory of Evidence*. Princeton University Press, 1976.

[16] T. Tervonen, R. Lahdelma, and P. Salminen. A method for elicitating and combining group preferences for stochastic multicriteria acceptability analysis. TUCS Technical Report 638, Turku Centre for Computer Science, Turku, Finland, November 2004.

[17] L.V. Utkin. Probabilities of judgments provided by unknown experts by using the imprecise Dirichlet model. *Risk, Decision and Policy*, 9(4):391 – 400, 2004.

[18] L.V. Utkin. Extensions of belief functions and possibility distributions by using the imprecise Dirichlet model. *Fuzzy Sets and Systems*, 154(3):413–431, 2005.

[19] L.V. Utkin. Ranking procedures by pairwise comparison using random sets and the imprecise Dirichlet model. *Applied Mathematics and Computations*, 183(1):394–408, 2006.

[20] L.V. Utkin and Th. Augustin. Decision making under incomplete data using the imprecise Dirichlet model. *International Journal of Approximate Reasoning*, 44(3):322–338, 2007.

[21] P. Walley. Inferences from multinomial data: Learning about a bag of marbles. *Journal of the Royal Statistical Society, Series B*, 58:3–57, 1996. with discussion.

[22] K. Weichselberger. *Elementare Grundbegriffe einer allgemeineren Wahrscheinlichkeitsrechnung*, volume I Intervallwahrscheinlichkeit als umfassendes Konzept. Physika, Heidelberg, 2001.