Reliability analysis of load-sharing m-out-of-n systems with arbitrary load and different probability distributions of time to failure

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Abstract: The reliability analysis of load-sharing m-out-of-n systems where the workload is shared by the remaining working units when a unit fails is proposed in the paper. General expressions are provided for the m-out-of-n system reliability by arbitrary probability distributions of time to failure of units. Simplified methods for computing the survivor function in cases when the time to unit failure is governed by the Weibull and exponential probability distributions. The system survivor function and the mean time to failure in the explicit form are obtained for systems with arbitrary load (decreasing and increasing) by the exponential time to unit failure. Numerical examples illustrate the properties of load-sharing m-out-of-n systems.

Keywords: reliability, load-sharing system, m-out-of-n system, survivor function, Weibull distribution, exponential distribution, mean time to failure, Monte Carlo technique.

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1. Introduction

Many systems with redundant units can be regarded as load-sharing systems. If a unit fails in the load-sharing system, the workload is shared by the remaining working units such that the load on these units increases. As pointed out by Huang and Xu (2010), many systems are load-sharing when a load is shared by several units, including electric networks, multiprocessor computer systems, hydraulic systems, mechanical systems, etc. One of the pioneering works devoted to load-share models applied in the textile industry was proposed by Daniels (1945). Daniels originally adopted the load-share model to describe how the strain on yarn fibers increases as individual fibers within a bundle break. A simple example of a load-sharing system in construction engineering is provided by Yang and Younis (2005) considering a structure consisting of two beams which are supporting a shared load. A clear example illustrating the impact of the shared load is provided by Kvam and Pena (2005). The authors of this work describe a large structure supported by welded joints. The structure fails only after a series of

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supporting joints fail. The failure of one or two welded joints in a bridge support, for instance, might cause the stress on remaining joints to increase, thus causing earlier subsequent failures.

Many real applications can be modelled by means of m-out-of-n systems. A m-out-of-n system fails after failures of arbitrary $m+1$ units, $m+1 \le n$. It is supposed that the system consists of $n-m$ working units and m redundant units. One of the system characteristics of redundancy is the redundancy rate which is $n/(n-m)$ in m-out-of-n systems.

A lot of methods of computing the system reliability without taking into account the shared load are available in literature (see, for example, Kuo and M.J. Zuo, 2003; Misra,1992). In particular, the survivor function of the system or the probability that the system is in the working state under condition of identical and independent units is defined as

$$P(t) = \sum_{i=0}^{m} C_n^i F^i(t) \overline{F}^{n-i}(t). \tag{1}$$

Here $F(t)$ is the probability of time to failure, $\overline{F}(t) = 1 - F(t)$ is the unit survivor function, $C_n^i = n!/(i!(n-i)!)$ is the binomial coefficient.

In order to take into account the fact that the working units share the load after some units fail, load-sharing m-out-of-n systems are studied by many authors (Amari et al., 2006; Amari and Bergman, 2008; Huang and Xu, 2010; Jain and Gupta, 2012; Liu, 1998; Mohammad, 2013; Qi et al., 2014; Shao and Lamberson,1991; Scheuer, 1998; Yinghui and Jing, 2008; Yun et al., 2012). The authors consider different conditions of m-out-of-n system functioning and different applications modelled by m-out-of-n systems.

In this paper, we provide general expressions for the m-out-of-n system reliability by arbitrary probability distributions of time to failure of units. Moreover, rather simple expressions for the survivor function in case when the time to unit failure is governed by the Weibull probability distribution are derived. We analyze the system reliability by exponentially distributed unit times to failure. The main contribution is explicit expressions for the case of arbitrary values of load by exponential distributions of time to unit failure. At that, it is supposed that the load can decrease, but not only increase after failures of units. This case has not studied in the literature. Numerical examples illustrate the properties of load-sharing m-out-of-n systems.

The paper is organized as follows. Section 2 provides definitions of m-out-of-n systems under shared load and considers a general algorithm for computing system reliability measures, for example, the survivor function. Simplified expressions for the survivor functions under condition of the Weibull probability distribution of time to unit failure are presented in Section 3. A way for obtaining very simple expressions for load-sharing m-out-of-n systems when their units are identical and their time to failure are governed by the exponential distribution is given in Section 4. The reliability measures of load-sharing m-out-of-n systems with arbitrary load are derived in Section 5.

2. The probability of system time to failure under the load

In load-sharing m-out-of-n systems, the failure rate of a unit is affected by the magnitude of load it shares. We suppose that the load k_i on the i-th unit of the m-out-of-n system under normal conditions is 1. Below we will say that the system is without load when $k_i = 1$. This does not mean that there is no load. We use this phrase for brevity.

Let us evaluate the system reliability taking into account the failure aftereffect or a change of the load on units after failure of some units. We assume that the load on the working units after failure of arbitrary i units increases and is equal to

$$k_i = \frac{n}{n-i}, \ i = 1,2,...,m. \tag{2}$$

If t_i is a time moment of the i-th unit failure, then the load function characterizing the degree of load on other units is of the form:

$$k(t)=\begin{cases}1, & if \quad t<t_1\\ \dfrac{n}{n-1}, & if \quad t_1\le t<t_2\\ \dfrac{n}{n-2}, & if \quad t_2\le t<t_3\\ \dots & \\ \dfrac{n}{n-m}, & if \quad t\ge t_m\end{cases}. \tag{3}$$

The survivor function of the m-out-of-n system under changes of the load can be written as

$$P_c(t)=p_0(t)+\sum_{i=1}^{m}\sum_{j_1,j_2,\dots,j_i}p_{j_1,j_2,\dots,j_i}(t),$$

where $p_0(t)$ is the probability that all units are in the working state at time t; $p_{j_1,j_2,\dots,j_i}(t)$ is the probability that exactly i units with numbers $j_1,j_2,\dots,j_i$ failed consequently with numbers $j_1,j_2,\dots,j_i$ before time t, i.e., the units with numbers $j_1,j_2,\dots,j_i$ failed at time moments $t_1<t_2<\dots<j_i\le t$, respectively. The summation is over all indices $j_1,j_2,\dots,j_i$ from 1 to m.

If the system consists of identical units, then we get a more simple expression

$$P_c(t)=p_0(t)+\sum_{i=1}^{m}A_n^i\,p_{1,2,\dots,i}(t). \tag{4}$$

Here $A_n^i=n!/(n-i)!$; $p_0(t)$ is the probability that all units are in the working state at time t; $p_{1,2,\dots,i}(t)$ is the probability that units with numbers $1,2,\dots,i$ consequently failed before time t.

Let us write the corresponding expressions for the above probabilities $p_0(t)$ and $p_{1,2,\dots,i}(t)$. The probability that all units are in the working state at time t can be computed as

$$p_0(t)=\prod_{i=1}^{n}\overline{F}_i(t). \tag{5}$$

The probability $p_{1,2,\dots,i}(t)$ can be computed by using the total probability formula

$$\begin{aligned}p_{1,2,\dots,i}(t)=&\int\dots\int_{t_1\le t_2\le\dots\le t_i\le t} f_1(t_1)f_2(t_2,t_1)f_3(t_3,t_1,t_2)\dots f_i(t_i,t_1,t_2,\dots,t_{i-1})\\ &\times\overline{F}_{i+1}(t,t_1,\dots,t_i)..\overline{F}_n(t,t_1,\dots,t_i)dt_1\dots dt_i\end{aligned} \tag{6}$$

We will use the following notations:

$f_1(t)$ is the probability density function (PDF) of time to failure of the first unit.

$f_2(t,t_1)$ is the PDF of time to failure of the second unit under condition that the first unit fails at time t_1. The load on units with numbers $2,3,\dots,n$ is changed at time t_1.

$f_3(t,t_1,t_2)$ is the PDF of time to failure of the third unit under condition that the first unit fails at time t_1, the second unit fails at time t_2. The load on units with numbers $3,\dots,n$ is changed twice at times t_1 and t_2.

$f_i(t,t_1,t_2,\dots,t_{i-1})$ is the PDF of time to failure of the i-th unit under condition that the first unit fails at time t_1, the second unit fails at time $t_2,\dots$, the unit with number $i-1$ fails at time t_{i-1}. The load on units with numbers $i,\dots,n$ is changed at times $t_1,t_2,..,t_{i-1}$.

The survivor functions $\overline{F}_1(t)$, $\overline{F}_2(t,t_1)$, $\overline{F}_3(t,t_1,t_2),\dots,\overline{F}_{i+1}(t,t_1,t_2,\dots,t_i),\dots,\overline{F}_n(t,t_1,t_2,\dots,t_i)$ have the same meaning. These probabilities also depend on the history of unit failures in the system.

Note that expressions (4)-(6) are valid for the arbitrary shared load on the working units caused by failures of some units of the system. This shared load can be computed by using (2).

A method for computing the above conditional PDFs and survivor functions is based on the results of the paper by Gurov and Utkin (2012). The reliability of systems under the piecewise constant load (3) by means of the so-called discrete load-share reliability models was studied in the paper. We give a brief description of these models below. An important assumption was accepted in the paper, namely, it was assumed that the load increases in k times as the system failure rate increases in k times. This assumption allows getting rather simple expressions for the survivor function under the load.

The variable t_i of the function $\overline{F}_j(t,t_1,t_2,...,t_i)$ is the time moment of the last failure such that a unit with number $i-1$ after this failure changes its load from value k_{i-1} on k_i. Since the inequality $t_i \le t$ is valid, then the conditional probability is computed through the unconditional probability as follows:

$$\overline{F}_j(t,t_1,t_2,...,t_i) = \overline{F}_j^{k_i}(t - x_i).$$

Here the bias parameter x_i is computed by means of the recurrent algorithm through $x_1, x_2,..., x_{i-1}$ from the following equations:

$$\overline{F}_j^{k_1}(t_1 - x_1) = \overline{F}_j(t_1),$$
$$\overline{F}_j^{k_2}(t_2 - x_2) = \overline{F}_j^{k_1}(t_2 - x_1),$$
$$\ldots$$
$$\overline{F}_j^{k_i}(t_i - x_i) = \overline{F}_j^{k_{i-1}}(t_i - x_{i-1}).$$

If we differentiate the probability $\overline{F}_j(t,t_1,t_2,...,t_i)$ with respect to t, then we get the expression for the conditional PDF through the unconditional PDF and the unconditional survivor function of the j-th unit as

$$f_j(t,t_1,t_2,...,t_i) = k_i \overline{F}_j^{k_i-1}(t - x_i) f_j(t - x_i).$$

In sum, expressions (4)-(6) allows us to compute the survivor function of the m-out-of-n system by arbitrary probability distributions of its unit time to failure. Main difficulties of the computation can be met by solving the algebraic equations and by computing multiple integrals.

The first difficulty can be avoided if to develop a library of the auxiliary software programs for computing values of survivor functions and inverse survivor functions. For computing the multiple integrals, the Monte Carlo simulation technique can be used. Random points $(x_1, x_2,..., x_i)$ uniformly distributed in the simplex $x_1 + x_2 + ... + x_i \le t$ are generated. An important question here is the number N of the generated points. The number N is taken in order to guaranty a predefined accuracy of computing the function $P(t)$ of the system without load. It can be carried out in two ways:

- by using (1);
- by using the Monte Carlo technique for the case of lack of the shared load on working units after failure of other units, i.e., when $k(t) \equiv 1$.

3. The Weibull probability distribution of time to failure

The method for computing the reliability of m-out-of-n systems under shared load proposed in the previous section can be simplified if to assume that the unit time to failure is governed by the Weibull probability distribution. We assume that the time to failure of the i-th unit has the Weibull distribution with parameters α_i and β_i. It has been shown by Gurov and Utkin (2014a) that there is a simple connection between conditional and unconditional survivor functions, which is of the form:

$$\overline{F}_j(t,t_1,...,t_i)=\overline{F}_j\left(\int_0^t k^{\frac{1}{\alpha_j}}(\tau)d\tau\right)=\overline{F}_j\left(t_1+k_1^{\frac{1}{\alpha_j}}(t_2-t_1)+...+k_i^{\frac{1}{\alpha_j}}(t-t_i)\right).$$

Here the load $k(\tau)$ is determined from (3).

Differentiating the above expression, we get the PDF of time to failure of the j-th unit under condition that units with numbers $1,2,...,i$ have failed. The PDF is

$$f_j(t,t_1,...,t_i)=k_i^{\frac{1}{\alpha_j}} f_j\left(t_1+k_1^{\frac{1}{\alpha_j}}(t_2-t_1)+...+k_i^{\frac{1}{\alpha_j}}(t-t_i)\right).$$

If we assume that all units in the system are identical, then we can write

$$p_0(t)=\overline{F}^{\,n}(t) \tag{7}$$

and

$$\begin{aligned}p_{1,2,...,i}(t)&=(k_1...k_{i-1})^{\frac{1}{\alpha}}\\&\times\int\limits_{t_1\le t_2\le...\le t_i\le t}\!\!...\!\int f(t_1)f\left(t_1+k_1^{\frac{1}{\alpha}}(t_2-t_1)\right)...f\left(t_1+k_1^{\frac{1}{\alpha}}(t_2-t_1)+...+k_{i-1}^{\frac{1}{\alpha}}(t_i-t_{i-1})\right)\\&\times\overline{F}^{\,n-i}\left(t_1+k_1^{\frac{1}{\alpha}}(t_2-t_1)+...+k_i^{\frac{1}{\alpha}}(t-t_i)\right)dt_1...dt_i\end{aligned} \tag{8}$$

So, if we know that the unit time to failure is governed by the Weibull probability distribution, then expressions (4), (7)-(8) give a simple way for computing the survivor function of the m-out-of-n system, which takes into account the shared load after failures of some units.

4. The exponential probability distribution of time to failure

If units of a m-out-of-n system are identical and their time to failure are governed by the exponential distribution with failure rate λ, then the reliability analysis of the system under the shared load is significantly simplified even in comparison with expressions (4), (7)-(8). In this case, the survivor function is of the form:

$$P_c(t)=\sum_{i=0}^{m}\frac{(n\lambda t)^i}{i!}e^{-n\lambda t}\ . \tag{9}$$

This means that the time to failure of the system has the Erlangen distribution with parameters $\alpha=m+1$ and $\beta=1/(n\lambda)$. In order to prove this property, we return to expressions (7) and (8). According to (7), we get

$$p_0(t)=e^{-n\lambda t}.$$

By taking in (8) $\alpha=1$, we obtain

$$p_{1,2,...,i}(t)=\lambda^i k_1...k_{i-1}\int\limits_{t_1\le t_2\le...\le t_i\le t}\!\!...\!\int e^{-\lambda S(t,t_1,...,t_i)}dt_1...dt_i\,.$$

Here

$$\begin{aligned}S(t,t_1,...t_i)=&\,t_1+(t_1+k_1(t_2-t_1))+...+(t_1+k_1(t_2-t_1)+...+k_{i-1}(t_i-t_{i-1}))+\\&(n-i)(t_1+k_1(t_2-t_1)+...+k_i(t-t_i))\end{aligned}.$$

After modification, we get

$$\begin{aligned}S(t,t_1,...t_i)&=nt_1+(n-1)k_1(t_2-t_1)\\&+...+(n-i+1)k_{i-1}(t_i-t_{i-1})+(n-i)k_i(t-t_i)\end{aligned}. \tag{10}$$

Due to (2), we can rewrite the above equality as follows:

$$S(t,t_1,...t_i)=nt_1+n(t_2-t_1)+...+n(t_i-t_{i-1})+n(t-t_i)=nt\,.$$

Hence, there holds

$$p_{1,2,\dots,i}(t) = \lambda^i \frac{n^{i-1}}{(n-1)\dots(n-i+1)} \int\limits_{t_1 \le t_2 \le \dots \le t_i \le t} \dots \int e^{-\lambda nt} dt_1 \dots dt_i$$

$$= \lambda^i \frac{n^{i-1}}{(n-1)\dots(n-i+1)} e^{-\lambda nt} \int\limits_{t_1 \le t_2 \le \dots \le t_i \le t} \dots \int dt_1 \dots dt_i .$$

It is obvious that the above multiple integral is equal to the volume of the i-dimensional simplex with edges of size t, i.e., it is

$$\int\limits_{t_1 \le t_2 \le \dots \le t_i \le t} \dots \int dt_1 \dots dt_i = \frac{t^i}{i!} .$$

Consequently, there holds

$$p_{1,2,\dots,i}(t) = \lambda^i \frac{n^{i-1} t^i}{(n-1)\dots(n-i+1) i!} e^{-\lambda nt} = \frac{(\lambda nt)^i (n-i)!}{n! i!} e^{-\lambda nt} = \frac{1}{A_n^i} \frac{(\lambda nt)^i}{i!} e^{-\lambda nt} .$$

Equality (4) shows that

$$P_c(t) = e^{-n\lambda t} + \sum_{i=1}^{m} \frac{(\lambda nt)^i}{i!} e^{-\lambda nt} .$$

We get (9), as was to be proved.

It is interesting to note that the survivor function $P_c(t)$ of the m-out-of-n system with shared load and units having failure rate λ coincides with the survivor function of the cold-standby system having one working unit and m redundant units in the idle state if the units have failure rate $n\lambda$.

By having simple expressions for the survivor function of the m-out-of-n system, we can compare the mean time to failure of the system without load T_1 and under the shared load T_{1c}. It follows from (1) and (9) that T_1 is

$$T_1 = \frac{1}{\lambda} \sum_{i=0}^{m} \frac{1}{n-i} C_n^i \sum_{j=0}^{m-i} (-1)^j C_{n-i}^j .$$

Since the expectation of the Erlangen distribution is the product of parameters α and β, then we can write

$$T_{1c} = \frac{m+1}{n\lambda} .$$

By computing the mean time to failure for different values of m and n, we can conclude that T_1 increases as the number of system units n and the number of redundant units m increase. At that, the difference $T_1 - T_{1c}$ between these measures also increases with m and n. For example, if $n = 6$, $m = 5$, then $T_1 = 2.45/\lambda$, $T_{1c} = 1/\lambda$. If $n = 31$, $m = 30$, then we have $T_1 = 4.03/\lambda$, $T_{1c} = 1/\lambda$. This means that the aftereffect in the first case reduces the mean time to failure approximately in 2.5 times, in the second case – in 4 times.

Example 1. Let us consider a m-out-of-n system with $n = 2$, $m = 1$ (a parallel system). The system units are identical and their times to failure have exponential distribution with failure rate λ. Let us compute the survivor functions $P(t)$, $P_c(t)$, the mean times to failure T_1, T_{1c} and compare them. We can write for the system without load

$$P(t) = \bar{F}^2(t) + 2F(t)\bar{F}(t).$$

It follows from (4) that the survivor function of the system with shared load is

$$P_c(t) = p_0(t) + 2p_1(t).$$

Probabilities $p_0(t)$ and $p_1(t)$ are computed from (5) and (6), respectively, i.e., we get

$$p_0(t) = \overline{F}^2(t),\ p_1(t) = \int_0^t f(t_1)\overline{F}(t,t_1)dt_1 .$$

Hence

$$P_c(t) = \overline{F}^2(t) + 2\int_0^t f(t_1)\overline{F}(t,t_1)dt_1 .$$

Since $\overline{F}(t) = e^{-\lambda t}$ for the exponential distribution, then we obtain
$\overline{F}(t,t_1) = \overline{F}(t_1 + 2(t - t_1)) = e^{-\lambda(2t-t_1)}$.

Consequently, we can write for the system without load

$$P(t) = e^{-2\lambda t} + 2(1 - e^{-\lambda t})e^{-\lambda t} = 2e^{-\lambda t} - e^{-2\lambda t}, \quad (11)$$

and for the system with shared load

$$P_c(t) = e^{-2\lambda t} + 2\int_0^t \lambda e^{-\lambda t_1} e^{-\lambda(2t-t_1)} dt_1 = (1 + 2\lambda t)e^{-2\lambda t} . \quad (12)$$

Let us take for example $\lambda = 0.1\ \mathrm{h}^{-1}$. The corresponding survivor functions computed from (11) and (12) are shown in Fig.1.

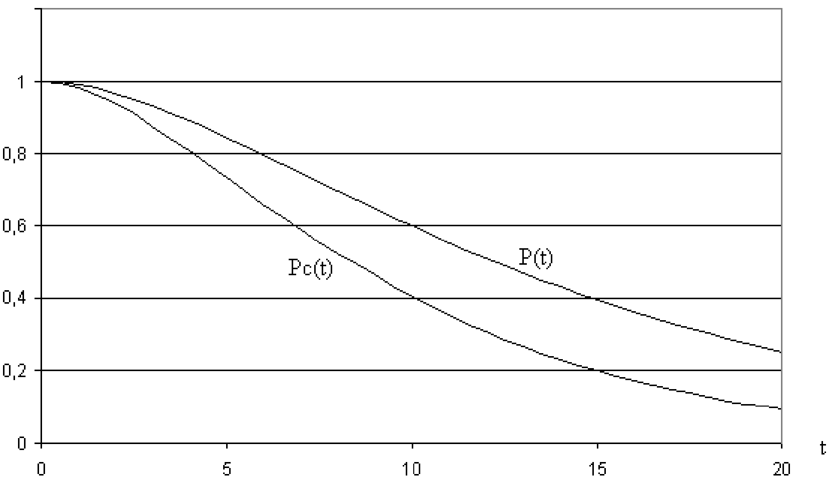

Figure 1. Survivor functions $P(t)$ and $P_c(t)$ for the 1-out-of-2 system

It follows from Fig.1 that taking into account the shared load significantly reduces the survivor function and, therefore, the system reliability. The largest difference of the function is observed at point $t = 12.5$ h and is 0.2. The relative difference increases with time. For example, it is 63% at time $t = 20$ h.

The mean time to failure of the system without load is

$$T_1 = \int_0^\infty (2e^{-\lambda t} - e^{-2\lambda t})dt = \frac{2}{\lambda} - \frac{1}{2\lambda} = \frac{3}{2\lambda} .$$

The mean time of the system with shared load is

$$T_{1c} = \int_0^\infty (1 + 2\lambda t)e^{-2\lambda t} dt = \frac{1}{2\lambda} + \frac{1}{2\lambda} = \frac{1}{\lambda} .$$

One can see that T_1 is larger than T_{1c} in 1.5 times.

The above numerical example is very simple and all resulting reliability measures can be obtained without exploiting special software programs. However, it cannot be done for most reliability applications. The next example illustrates this computational difficulty.

The algorithm of computing the reliability measures of m-out-of-n systems with the shared load where units are governed by the Weibull distribution is used in the next example. It should be noted that the use of (4), (7), (8) is simpler than a general case of m-out-of-n systems, but even in this case we have to apply a special software program.

Example 2. Let us consider a 6-out-of-8 system ($n = 8$, $m = 6$), consisting of identical units whose time to failure has the Weibull distribution with the mean value 15 h and the standard deviation 3 h. Results of applying a numerical algorithm for computing the survivor functions $P_c(t)$ and $P(t)$ are provided in Fig. 2. The algorithm computes the multiple integrals by using the Monte Carlo technique with $N = 300\ 000$ trials. The number of trials is determined such that the value of $P(t)$ computed from (1) coincide with values of $P_c(t)$ computed from (4)-(6) when the load function is 1, i.e., $k(t) = 1$. It can be seen from Fig. 2 that the functions differ from each other, for instance, the probability of time to failure of the system without load is 0.67 at time 17 h whereas the probability of time to failure of the system under the shared load is 0.48. This difference shows the importance of taking into account real conditions of the system functioning, in particular, the conditions of the shared load. The corresponding mean times to failure are $T_1 = 17.6$ and $T_{1c} = 16.9$.

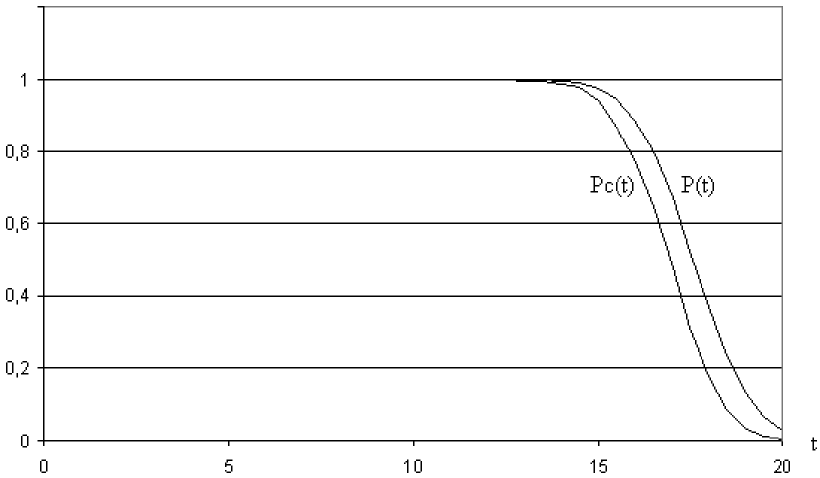

Figure 2. Survivor functions $P_c(t)$ and $P(t)$ for the 6-out-of-8 system

5. Systems with arbitrary load

So far we have considered the case when the shared load increases. However, the conditions of the system behavior may be different, i.e., the reliability may increase. This can be done by reducing the unit failure rate in case of the hot-standby redundancy. This type of redundancy a priori considers a possibility of working under the higher load in the initial state before failures.

Let us consider a m-out-of-n system under conditions of decreasing the load on the system units as well as its increasing. In the first case, the load rate k_i is less than 1 (the hot-standby system). In the second case, the load rate is larger than 1.

If the probability distributions of the unit time to failure are exponential, then we can derive rather simple explicit expressions for computing the reliability measures of systems with identical units. First, we derive the expression for computing the following multiple integral:

$$p_i(t) = \int\limits_{t_1 \le t_2 \le \ldots \le t_i \le t} \ldots \int e^{-(c_1 t_1 + c_2 t_2 + \ldots + c_i t_i)} dt_1 \ldots dt_i, \; i = 1,2,\ldots,m.$$

It is obvious that it can be rewritten in the form of the following recurrence formula:

$$p_i(t) = \int_0^t e^{-c_i t_i} p_{i-1}(t_i) dt_i. \qquad (13)$$

By applying the Laplace transform, we rewrite (13) as

$$\hat{p}_i(z) = \frac{1}{z}\hat{p}_{i-1}(z + c_i).$$

By using a similar equality for the function $\hat{p}_{i-1}$, we get

$$\hat{p}_i(z) = \frac{1}{z(z + c_i)}\hat{p}_{i-2}(z + c_i + c_{i-1}).$$

Repeating consequently this procedure, we obtain the Laplace representation of the multiple integral $p_i(t)$ in the form of the rational fraction:

$$\hat{p}_i(z) = \frac{1}{z(z + c_i)(z + c_i + c_{i-1})\ldots(z + c_i + c_{i-1} + \ldots + c_1)}. \qquad (14)$$

Let us suppose that all roots of the denominator are prime numbers. This implies that any consecutive sum of coefficients c_j is not zero, namely,

$$c_{j_1+1} + \ldots + c_{j_2} \neq 0, \text{ for } 0 \leq j_1 < j_2 \leq i. \qquad (15)$$

It is well known that function (14) can be represented as a sum of partial fractions

$$\hat{p}_i(z) = \frac{a_0^{(i)}}{z} + \frac{a_i^{(i)}}{z + c_i} + \frac{a_{i-1}^{(i)}}{z + c_i + c_{i-1}} + \ldots + \frac{a_1^{(i)}}{z + c_i + c_{i-1} + \ldots + c_1},$$

whose coefficients are expressed as

$$a_0^{(i)} = \lim_{z \to 0} z\hat{p}_i(z)$$

and

$$a_j^{(i)} = \lim_{z \to -(c_i + \ldots + c_j)} (z + c_i + .. + c_j)\hat{p}_i(z), \; j = 1,2,\ldots,i.$$

By applying the inverse Laplace transform, we obtain the following simple expression for the multiple integral:

$$p_i(t) = a_0^{(i)} + a_i^{(i)}e^{-c_i t} + a_{i-1}^{(i)}e^{-(c_i + c_{i-1})t} + \ldots + a_1^{(i)}e^{-(c_i + c_{i-1} + \ldots + c_1)t}. \qquad (16)$$

Now we find a way for computing the probabilities $p_{1,2,\ldots,i}(t)$ for the case of arbitrary load rate k_i. It follows from (10) that there holds

$$S(t, t_1, \ldots t_i) = (n - (n-1)k_1)t_1 + ((n-1)k_1 - (n-2)k_2)t_2 + \ldots$$
$$+ ((n - i + 1)k_{i-1} - (n - i)k_i)t_i + (n - i)k_i t,$$

or

$$S(t, t_1, \ldots t_i) = \sum_{j=1}^{i} ((n - j + 1)k_{j-1} - (n - j)k_j)t_j + (n - i)k_i t.$$

Hence

$S(t, t_1, \ldots t_i) = \sum_{j=1}^{i} c_j t_j + c_0 t$ is a linear function with coefficients $c_j = (n - j + 1)k_{j-1} - (n - j)k_j$ and $c_0 = (n - i)k_i$.

Let us assume that these coefficients c_j satisfy inequalities (15), that is

$$c_{j_1+1} + \ldots + c_{j_2} = (n - j_1)k_{j_1} - (n - j_2)k_{j_2} \neq 0.$$

This means that expansion of probabilities $p_i(t)$ in the form of (16) can be used only when the load rates satisfy the following condition:

$$\frac{k_{j_2}}{k_{j_1}} \neq \frac{n - j_1}{n - j_2}.$$

Since

$$p_{1,2,\ldots,i}(t) = \lambda^i k_1 \ldots k_{i-1} \int\ldots\int_{t_1 \le t_2 \le \ldots \le t_i \le t} e^{-\lambda S(t,t_1,\ldots,t_i)} dt_1 \ldots dt_i,$$

then we obtain due to (16)

$$p_{1,2,\ldots,i}(t) = k_1 \ldots k_{i-1} e^{-\lambda c_0 t} \left(a_0^{(i)} + \sum_{j=1}^{i} a_j^{(i)} e^{-\lambda (c_i + \ldots + c_{i-j+1}) t} \right).$$

It follows from the above and from (4) that the final expression for computing the survivor function is of the form:

$$P_c(t) = e^{-\lambda n t} + \sum_{i=1}^{m} A_n^i k_1 \ldots k_{i-1} e^{-\lambda c_0 t} \left(a_0^{(i)} + \sum_{j=1}^{i} a_j^{(i)} e^{-\lambda (c_i + \ldots + c_{i-j+1}) t} \right). \quad (17)$$

We can use (17) for computing the system reliability measures by arbitrary load except for some special cases considered above. For example, the case of the form (2) cannot be studied by using (17). However, this case has been considered for the survivor function expressed by (9).

We find also the mean time to failure on the basis of (17) as follows:

$$T_{1c} = \frac{1}{n\lambda} + \frac{1}{\lambda} \sum_{i=1}^{m} A_n^i k_1 \ldots k_{i-1} \left(\frac{a_0^{(i)}}{c_0} + \sum_{j=1}^{i} \frac{a_j^{(i)}}{c_0 + c_i + \ldots + c_{i-j+1}} \right). \quad (18)$$

Example 3. Let us consider the 6-out-of-8 system ($n = 8$, $m = 6$), consisting of identical units whose time to failure has the exponential distribution with the failure rate $\lambda = 0.1\ \mathrm{h}^{-1}$. The system can work in two regimes:

- the load rate on the working units after failures of i units is $m/(n-i)$, $i = 1,2,\ldots,m$;
- the load rate on the working units after the first failure decreases in 20% and becomes equal to $0,8$ (the hot-standby system).

Our aim is to compute the reliability measures of the system without load ($k = 1$) and under the shared load by its increasing ($k > 1$) and decreasing ($k < 1$).

The curves of the corresponding survivor functions are shown in Fig.3.

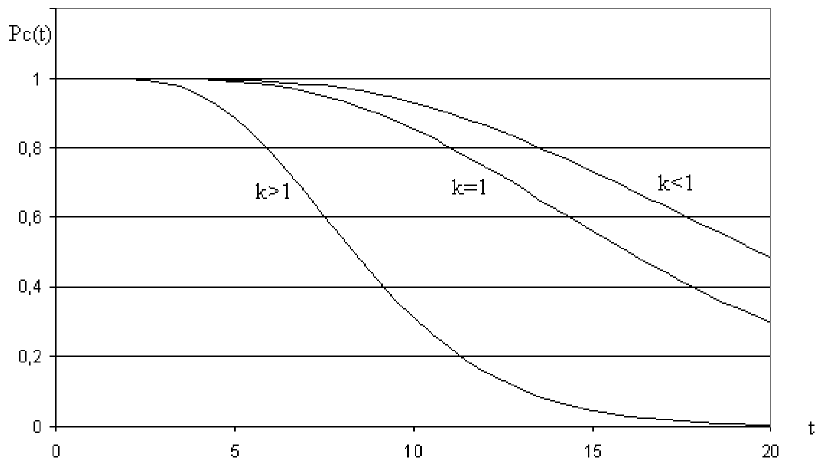

Figure 3. Survivor functions for the 6-out-of-8 system by different k

This example clearly illustrates how strongly the type of load impacts on the system reliability. This impact can be also seen from the values of mean times to failure, which $T_1 = 17.18$ h by $k = 1$, $T_{1c} = 8.75$ h by $k > 1$, and $T_{1c} = 21.16$ h by $k < 1$.

6. Conclusion

The reliability analysis of load-sharing m-out-of-n systems under the shared load by different conditions for the load and different probability distributions of time to unit failure has been provided in the paper. A special case of the Weibull probability distribution of time to unit failure has been considered in detail. In this case, the reliability measures, for instance, the survivor function, can be obtained in a simplified way. Similar expressions for the exponential time to unit failure have been derived in the explicit form. An interesting fact concerning the parallel and cold-standby systems has been observed. The reliability of a m-out-of-n system working under load and having units with failure rate λ coincides with the reliability of a cold-standby system if its units have failure rate $n\lambda$.

The most interesting case when the load is less than 1 has been considered in detail. It is important case because numerical examples show that the system reliability strongly depends on the shared load.

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