Imprecise reliability of cold standby systems

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Abstract: Most methods of reliability analysis of cold standby systems assume that the precise probability distributions of the component times to failure are available. However, this assumption may be unreasonable in a wide scope of cases (software, human-machine systems). Therefore, the imprecise reliability models of cold standby systems are proposed in the paper. These models suppose that arbitrary probability distributions of the component time to failure are possible and they are restricted only by available information in the form of lower and upper probabilities of some events. It is shown how the reliability assessments may vary with a type of available information. The impact of the independence condition on reliability of systems is studied. Numerical examples illustrate the proposed models.

Keywords: reliability, cold standby system, imprecise probability theory, possibility measure, probability distribution, independence

Introduction

The cold-standby systems have been discussed extensively in the literature (Kumar & Agarwal 1980). Most methods of reliability analysis of such systems assume that the precise probability distributions of the component times to failure are available. However, this assumption may be unreasonable in a wide scope of cases (software, human-machine systems) or may be violated. The reliability assessments, that are combined to describe a system and components, may come from various sources. Some assessments are based on relative frequencies or on well established statistical models. A part of the reliability assessments may be supplied by experts. Assessments may be also provided by a user of the system during the experimental service. In order to compute new reliability characteristics, to make decisions, and to use maximally available information, all these assessments need to be combined. To solve the problem of some incompleteness of available information, Kai-Yuan Cai (Cai 1996) has proposed to use the possibility measure in place of the probability one. Reliability analysis of a cold standby system whose failure behavior is fully characterized in the context of possibility measures (Dubois & Prade 1988) has been considered in (Cai, Wen & Zhang 1995). However, the possibility measure does not cover all possible types of partial information.

To cope with the problem of usage the heterogeneous and partial information, the theory of imprecise probabilities (also called the theory of lower previsions (Walley 1991), the theory of interval statistical models (Kuznetsov 1991), the theory of interval probabilities (Weichselberger 2000, Weichselberger 2001)) can be successfully applied. A general framework for the theory of imprecise probabilities is provided by upper and lower previsions. They can model a very wide variety of kinds of uncertainty, partial information, and ignorance. The rules used in the theory of imprecise probabilities, which are based on a general procedure called natural extension (optimization), can be applied to various measures.

The imprecise reliability models of various systems have been considered in the literature (Utkin & Gurov 1999, Gurov & Utkin 1999, Kozine & Filimonov 2000, Utkin & Gurov 2001). The reliability of cold standby systems under partial information about probabilities of times to failure of the system components is analyzed in this paper.

Problem statement

Each component of an unrepairable n-component cold standby system may have three states: operating, idle, and failed. In the operating state, the component performs its assigned functions. In its idle state, the component is operative,

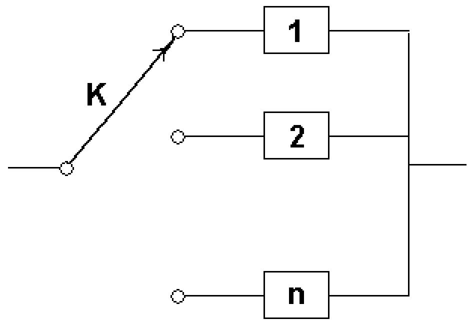

Figure 1: An example of the cold standby system

but does nothing, and no performance deterioration is possible. In its failed state, the component is non-operative. At any time, only one operative component is required and other components are redundant. The system is initiated with component 1 being in the operating state and other components are in idle states (see Fig.1). A failed component is immediately replaced by a redundant component through a conversion switch K with negligible time. Suppose all components are activated sequentially in order. A system failure occurs when no operative components are available. Let X_i be the time to failure of the i-th component, $i = 1, ..., n$. If we assume that the conversion switch is absolutely reliable, then the system time to failure is determined as $X_1 + ... + X_n$.

Let $\varphi_{ij}(X_i)$ be a function of the random time to failure X_i of the i-th component. According to (Barlow & Proschan 1975), the system lifetime can be uniquely determined by the component lifetimes. Suppose that information about n components is represented as a set of lower and upper previsions (expectations) $\underline{\mathbb{E}}\varphi_{ij}(X_i)$ and $\overline{\mathbb{E}}\varphi_{ij}(X_i)$, $i = 1, ..., n$, $j = 1, ..., m_i$, of functions φ_{ij}. Here m_i is a number of judgements that are related to the i-th component reliability. For example, if the lower and upper probabilities, $\underline{p}$ and $\overline{p}$, of the i-th component failure in an interval $[b, c]$ are available, then $\varphi_{ij}(X_i) = I_{[b,c]}(X_i)$ and $\underline{\mathbb{E}}I_{[b,c]}(X_i) = \underline{p}$, $\overline{\mathbb{E}}I_{[b,c]}(X_i) = \overline{p}$. Here $I_{[b,c]}(X)$ is the indicator function such that $I_{[b,c]}(X) = 1$ if $X \in [b, c]$ and $I_{[b,c]}(X) = 0$ if $X \notin [b, c]$. If we know the lower and upper mean time to failure, $\underline{T}$ and $\overline{T}$, of the i-th component, then $\varphi_{ij}(X_i) = X_i$ and $\underline{\mathbb{E}}X_i = \underline{T}$, $\overline{\mathbb{E}}X_i = \overline{T}$. In this case, the optimization problems for computing the lower and upper expectations of the system function g are

$$\underline{\mathbb{E}}g = \min_{\mathcal{P}} \int_{\mathbb{R}^n_+} g(x_1 + ... + x_n)\rho(x_1, ..., x_n)\mathrm{d}x_1 \cdots \mathrm{d}x_n, \tag{1}$$

$$\overline{\mathbb{E}}g = \max_{\mathcal{P}} \int_{\mathbb{R}^n_+} g(x_1 + ... + x_n)\rho(x_1, ..., x_n)\mathrm{d}x_1 \cdots \mathrm{d}x_n, \tag{2}$$

subject to

$$\int_{\mathbb{R}^n_+} \rho(x_1, ..., x_n)\mathrm{d}x_1 \cdots \mathrm{d}x_n = 1,\ \rho(x_1, ..., x_n) \geq 0,$$

$$\underline{\mathbb{E}}\varphi_{ij}(X_i) \leq \int_{\mathbb{R}^n_+} \varphi_{ij}(x_i)\rho(x_1, ..., x_n)\mathrm{d}x_1 \cdots \mathrm{d}x_n \leq \overline{\mathbb{E}}\varphi_{ij}(X_i),\ i \leq n, j \leq m_i. \tag{3}$$

Here the minimum and maximum are taken over the set $\mathcal{P}$ of all possible n-dimensional joint density functions $\{\rho(\mathbf{X})\}$ of the component times to failure satisfying conditions (3). The function g has the same sense as the functions φ_{ij}. Solutions to optimization problems (1)-(3) are defined on the set $\mathcal{P}$ of possible densities that are consistent with partial information expressed in the form of constraints (3).

It should be noted that only joint densities are used in optimization problems (1)-(3) because, in a general case, we may not be aware whether the variables $X_1,...,X_n$ are dependent or not. If it is known that components are independent, then $\rho(x_1,...,x_n) = \rho_1(x_1)\cdots\rho_n(x_n)$. In this case, the set $\mathcal{P}$ is reduced and consists only of the densities that can be represented as a product. As a result, we obtain a more narrow interval of $\underline{\mathbb{E}}g$ and $\overline{\mathbb{E}}g$. The optimization problems for computing new lower and upper expectations are of the form:

$$\underline{\mathbb{E}}g = \min_{\mathcal{P}} \int_{\mathbb{R}_+^n} g(x_1+...+x_n)\rho_1(x_1)\cdots\rho_n(x_n)\mathrm{d}x_1\cdots\mathrm{d}x_n, \tag{4}$$

$$\overline{\mathbb{E}}g = \max_{\mathcal{P}} \int_{\mathbb{R}_+^n} g(x_1+...+x_n)\rho_1(x_1)\cdots\rho_n(x_n)\mathrm{d}x_1\cdots\mathrm{d}x_n, \tag{5}$$

subject to

$$\rho_i(x_i) \geq 0,\ \int_{\mathbb{R}_+} \rho_i(x_i)\mathrm{d}x_i = 1,$$

$$\underline{\mathbb{E}}\varphi_{ij}(X_i) \leq \int_{\mathbb{R}_+} \varphi_{ij}(x_i)\rho_i(x_i)\mathrm{d}x_i \leq \overline{\mathbb{E}}\varphi_{ij}(X_i),\ i\leq n,\ j\leq m_i. \tag{6}$$

Example 1 *Let us consider a two-component cold-standby system. The following information about reliability of components is available:*

Component 1: the probability of failure before 8 *hours is less than* 0.01*, the mean time to failure is* 26 *hours;*

Component 2: the probability of failure after 3 *hours is between* 0.98 *and* 0.99*.*

Let us find the probability of the system failure before time 10 *hours.*

In this case, there are two judgements, $m_1 = 2$*, about the first component and one judgement,* $m_2 = 1$*, about the second component. The formal representation of judgements is* $\varphi_{11}(X_1) = I_{[0,8]}(X_1)$, $\varphi_{12}(X_1) = X_1$, $\varphi_{21}(X_2) = I_{[3,\infty)}(X_2)$, $g(X_1,X_2) = I_{[0,10]}(X_1+X_2)$, $\underline{\mathbb{E}}\varphi_{11}(X_1) = 0$, $\overline{\mathbb{E}}\varphi_{11}(X_1) = 0.01$, $\underline{\mathbb{E}}\varphi_{12}(X_1) = \overline{\mathbb{E}}\varphi_{12}(X_1) = 26$, $\underline{\mathbb{E}}\varphi_{21}(X_2) = 0.98$, $\overline{\mathbb{E}}\varphi_{21}(X_2) = 0.99$. *By assuming that the system components are independent, optimization problems (4)-(6) can be rewritten as*

$$\underline{\mathbb{E}}g\ \left(\overline{\mathbb{E}}g\right) = \min_{\mathcal{P}}(\max_{\mathcal{P}}) \int_{\mathbb{R}_+^2} I_{[0,10]}(x_1+x_2)\rho_1(x_1)\rho_2(x_2)\mathrm{d}x_1\mathrm{d}x_2,$$

subject to

$$\rho_i(x_i) \geq 0,\ \int_{\mathbb{R}_+} \rho_i(x_i)\mathrm{d}x_i = 1,\ i=1,2,$$

$$0 \leq \int_{\mathbb{R}_+} I_{[0,8]}(x_1)\rho_1(x_1)\mathrm{d}x_1 \leq 0.01,$$

$$26 \leq \int_{\mathbb{R}_+} x_1\rho_1(x_1)\mathrm{d}x_1 \leq 26,$$

$$0.98 \leq \int_{\mathbb{R}_+} I_{[3,\infty)}(x_2)\rho_2(x_2)\mathrm{d}x_2 \leq 0.99.$$

Hence the lower and upper probabilities of the system failure before time 10 *hours, obtained as numerical solutions to the above optimization problems, are* 0 *and* 0.026*.*

Example 1 shows that it is necessary to solve non-linear optimization problems for computing the bounds for the system reliability. In case of a large number of components and corresponding judgements about their functioning, optimization problems are extremely difficult. Therefore, the main aim of the paper is to find simple solutions to such types of problems for the most important special cases. At that, the calculated reliability measure is the probability $R(t)$ of the system failure before time t, i.e.,

$$R(t) = \Pr\{X_1 + ... + X_n \leq t\} = \mathbb{E}I_{[0,t]}(X_1 + ... + X_n).$$

This measure is called the unreliability. The reliability $Q(t)$ can be found as

$$Q(t) = \Pr\{X_1 + ... + X_n \geq t\} = \mathbb{E}I_{[t,\infty)}(X_1 + ... + X_n) = 1 - R(t).$$

If the system reliability measures are interval-valued, then $\overline{Q}(t) = 1 - \underline{R}(t)$ and $\underline{Q}(t) = 1 - \overline{R}(t)$.

It is worth noticing that the proposed approach for computing the interval reliability measures differs from the well-known interval analysis in which the uniform distribution inside intervals is assumed. Here it is supposed that arbitrary probability distributions are possible and they are restricted only by available information in the form of lower and upper previsions. The main advantage of the approach is that we do not introduce any additional assumptions concerning probability distributions which may lead to incorrect results.

Partially known probability distributions

Assume that the initial information about the time to failure of the i-th component is given in the following form:

$$\underline{p}_{ij} \leq \Pr\{X_i \leq \alpha_{ij}\} \leq \overline{p}_{ij},\ j = 1, ..., m_i, \tag{7}$$

and $\forall k \leq m_i$, $\forall j \leq m_i$ and $k \leq j$, the inequalities $\underline{p}_{ik} \leq \underline{p}_{ij}$ and $\overline{p}_{ik} \leq \overline{p}_{ij}$ are valid for all $i = 1, ..., n$. It is also assumed that

$$\alpha_{i1} \leq \alpha_{i2} \leq ... \leq \alpha_{im_i}.$$

This assumption is obvious because $\underline{p}_{ij}$, $\overline{p}_{ij}$, $j = 1, ..., m_i$, are the values of interval probability distributions. In other words, only m_i points of the probability distribution of X_i, $i = 1, ..., n$, are known with some accuracy. It should be noted that a lot of possible distributions can satisfy the above information. The illustration of the special case, when $\underline{p}_{ij} = \overline{p}_{ij} = p_{ij}$, is shown in Fig.2.

Independent components

Suppose the variables X_i, $i = 1, ..., n$, are independent. Then optimization problems (1)-(3) can be rewritten as

$$\underline{R}(t) = \min_{\mathcal{P}} \int_{\mathbb{R}_+^n} I_{[0,t]}(x_1 + ... + x_n)\rho_1(x_1)\cdots\rho_n(x_n)\mathrm{d}x_1\cdots\mathrm{d}x_n, \tag{8}$$

$$\overline{R}(t) = \max_{\mathcal{P}} \int_{\mathbb{R}_+^n} I_{[0,t]}(x_1 + ... + x_n)\rho_1(x_1)\cdots\rho_n(x_n)\mathrm{d}x_1\cdots\mathrm{d}x_n, \tag{9}$$

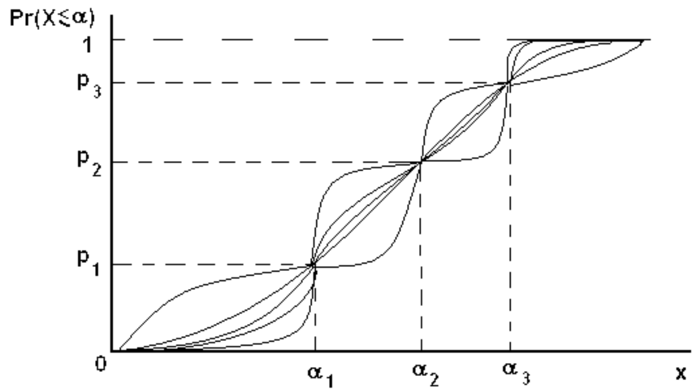

Figure 2: Illustration of constraints for the probability distributions

subject to

$$\int_{\mathbb{R}_+} \rho_i(x)\mathrm{d}x = 1,\ \rho_i(x) \geq 0,\ i = 1, ..., n,$$

$$\underline{p}_{ij} \leq \int_{\mathbb{R}_+} I_{[0,\alpha_{ij}]}(x)\rho_i(x)\mathrm{d}x \leq \overline{p}_{ij},\ i = 1, ..., n,\ j = 1, ..., m_i. \tag{10}$$

Without loss in generality, it is assumed $\underline{p}_{i0} = \overline{p}_{i0} = 0$, $\underline{p}_{i(m_i+1)} = \overline{p}_{i(m_i+1)} = 1$, $\alpha_{i0} = 0$, $\alpha_{i(m_i+1)} \to \infty$.

Introduce the following notation:

$$V = \left\{ (v_1, ..., v_n) : \sum_{i=1}^{n} \alpha_{iv_i} \leq t,\ v_i \in \{1, ..., m_i + 1\} \right\},$$

$$W = \left\{ (w_1, ..., w_n) : \sum_{i=1}^{n} \alpha_{i(w_i-1)} \geq t,\ \ w_i \in \{1, ..., m_i + 1\} \right\},$$

$$S = \{(s_1, ..., s_n) : \alpha_{is_i} \geq t,\ s_i \in \{1, ..., m_i + 1\}\},$$

Proposition 1 *If the system components are statistically independent and governed by partially known probability distributions in the form* $\Pr\{X_i \leq \alpha_{ij}\} = p_{ij}$ *and* $\alpha_{i1} \leq \alpha_{i2} \leq ... \leq \alpha_{im_i}$, $p_{i1} \leq p_{i2} \leq ... \leq p_{im_i}$, $i = 1, ..., n$, $j = 1, ..., m_i$, *then the lower and upper bounds for the unreliability of a cold standby system at time* t *are computed as follows:*

$$\underline{R}(t) = \sum_{V} \prod_{i=1}^{n} (p_{iv_i} - p_{i(v_i-1)}),\ \overline{R}(t) = 1 - \sum_{W} \prod_{i=1}^{n} (p_{iw_i} - p_{i(w_i-1)}).$$

Corollary 1 *If the system components are statistically independent and the probability distributions* $F_i(t) = \Pr(X_i \leq t)$ *of their times to failure are known precisely, then*

$$\underline{R}(t) = \overline{R}(t) = \int_0^t f_1 * ... * f_n(x)\mathrm{d}x.$$

Here $f_i(x)$ *is the probability density function of the random variable* X_i, $f_1 * ... * f_n(x)$ *is the convolution of densities.*

Corollary 1 states that the obtained expressions coincide with the conventional ones known in the reliability theory and this means that Proposition 1 generalizes conventional formulas for the unreliability of cold standby systems to the interval-valued unreliability.

Unfortunately, it is impossible, without analyzing a special system, to determine the dependency of $\underline{R}(t)$ and $\overline{R}(t)$ on the values $\underline{p}_{ij}$, $\overline{p}_{ij}$ if the information is represented in the form of (7) because a cold standby system is non-monotone. In this case, we can write

$$\underline{R}(t) = \min_{\underline{p}_{iv_i} \leq p_{iv_i} \leq \overline{p}_{iv_i}} \sum_{V} \prod_{i=1}^{n} (p_{iv_i} - p_{i(v_i-1)}),$$

$$\overline{R}(t) = \max_{\underline{p}_{iw_i} \leq p_{iw_i} \leq \overline{p}_{iw_i}} \left(1 - \sum_{W} \prod_{i=1}^{n} (p_{iw_i} - p_{i(w_i-1)}) \right).$$

Lack of knowledge about independence

We assumed in the previous section that the system components are independent. Now we remove this additional assumption and suppose that there is no information about independence of components.

The asterisk notation in $\underline{R}^*$ and $\overline{R}^*$ will mean that bounds for the unreliability are obtained based on the lack of information about independence of components.

Proposition 2 *If the system components are not judged to be independent, then the lower and upper bounds for the unreliability of a cold standby system at time t are computed as follows:*

$$\underline{R}^*(t) = \max_{V} \max \left\{ \sum_{i=1}^{n} \underline{p}_{iv_i} - (n-1), 0 \right\},$$

$$\overline{R}^*(t) = \min \left\{ \min_{S} \min_{i=1,\dots,n} \overline{p}_{is_i}, \min_{W} \min \left(1, \sum_{i=1}^{n} \overline{p}_{i(w_i-1)} \right) \right\}.$$

Corollary 2 *If there is no information about independence of the system components and the probability distributions $F_i(t) = \Pr(X_i \leq t)$ of the component times to failures are known precisely, then*

$$\underline{R}^*(t) = \max_{x_1+x_2+\dots+x_n=t} \max \left\{ \sum_{i=1}^{n} F_i(x_i) - (n-1), 0 \right\},$$

$$\overline{R}^*(t) = \min \left\{ \min_{x_1+x_2+\dots+x_n=t} \min_{i=1,\dots,n} F_i(x_i), \min_{x_1+x_2+\dots+x_n=t} \min \left(1, \sum_{i=1}^{n} F_i(x_i) \right) \right\}.$$

This means, even though the probability distributions of the component times to failure are known precisely and the judgement of the component independence is not introduced, then only imprecise reliability measures can be found.

Example 2 *Let us consider a two-component cold-standby system with identical components. Suppose that experts provided 5%, 50%, and 95% quantiles of an unknown probability distribution of the component time to failure: 5 days, 70 days, and 300 days. The assessments of the experts can be represented as follows:*

$$\Pr(X \leq 5) = 0.05,$$

$$\Pr(X \leq 70) = 0.5,$$

$$\Pr(X \leq 300) = 0.95.$$

It is necessary to compute bounds for the reliability of the system at time 50 days, i.e., we have to find $\overline{Q}(50) = 1 - \underline{R}(50)$ and $\underline{Q}(50) = 1 - \overline{R}(50)$. By using notation introduced in this section, we can write

$$\begin{aligned}
&\alpha_{11} = \alpha_{21} = 5,\ \alpha_{12} = \alpha_{22} = 70,\ \ \alpha_{13} = \alpha_{23} = 300,\\
&p_{11} = p_{21} = 0.05,\ p_{12} = p_{22} = 0.5,\ \ p_{13} = p_{23} = 0.95,\\
&p_{14} = p_{24} = 1.
\end{aligned}$$

Let us construct the sets $V = \{(v_1, v_2)\}$ and $W = \{(w_1, w_2)\}$:

$$\begin{aligned}
V &= \{(1,1)\},\\
W &= \{(2,3), (2,4), (3,2), (3,3), (3,4), (4,2), (4,3), (4,4)\}.
\end{aligned}$$

1. *Components are independent. By using Proposition 1, we find $\underline{R}(50)$ and $\overline{R}(50)$ as follows:*

$$\begin{aligned}
\underline{R}(50) &= (p_{11} - p_{10})(p_{21} - p_{20}) = 0.0025,\\
\overline{R}(50) &= 1 - (p_{12} - p_{11})(p_{23} - p_{22}) - (p_{12} - p_{11})(p_{24} - p_{23})\\
&\quad - (p_{13} - p_{12})(p_{22} - p_{21}) - (p_{13} - p_{12})(p_{23} - p_{22})\\
&\quad - (p_{13} - p_{12})(p_{24} - p_{23}) - (p_{14} - p_{13})(p_{22} - p_{21})\\
&\quad - (p_{14} - p_{13})(p_{23} - p_{22}) - (p_{14} - p_{13})(p_{24} - p_{23})\\
&\quad = 1 - 0.7 = 0.3.
\end{aligned}$$

   Hence $\underline{Q}(50) = 0.7$ and $\overline{Q}(50) = 0.9975$.

2. *Lack of knowledge about independence of components. By using Proposition 2, we can find*

$$\begin{aligned}
\underline{R}^*(50) &= \max\{p_{11} + p_{21} - 1, 0\} = 0,\\
\overline{R}^*(50) &= \min\{\min(p_{12}, p_{23}), \min(p_{12}, p_{24}), \min(p_{13}, p_{22}),\\
&\quad \min(p_{13}, p_{23}), \min(p_{13}, p_{24}), \min(p_{14}, p_{22}),\\
&\quad \min(p_{14}, p_{23}), \min(p_{14}, p_{24})\}\\
&\quad = 0.5.
\end{aligned}$$

   Hence $\underline{Q}^*(50) = 0.5$ and $\overline{Q}^*(50) = 1$.

   It should be noted that the assessments provided by the experts can be regarded as points of the exponential probability distribution of time to failure with the failure rate 0.01. Let us find, for comparing results, the system reliability under the assumption that the precise exponential distribution of the component time to failure is available. By using Corollary 1, we get for independent components:

$$\begin{aligned}
\underline{Q}(50) &= \overline{Q}(50)\\
&= 1 - \int_0^{50} \int_0^{y} (0.01)^2 \exp(-0.01x) \cdot \exp(-0.01(y - x)) \mathrm{d}x \mathrm{d}y\\
&= 0.01 \cdot 50 \cdot \exp(-0.01 \cdot 50) + \exp(-0.01 \cdot 50)\\
&= 0.91.
\end{aligned}$$

By using Corollary 2, we have for the case of the lack of knowledge about independence of components:

$$
\begin{aligned}
\underline{R}^{*}(50) &= \max_{x\geq 0} \max\left\{1-\exp(0.01\cdot x)-\exp(-0.01(50-x)),0\right\} = 0,\\
\overline{R}^{*}(50) &= \min\{\min_{x\geq 0} \min\left(1-\exp(0.01\cdot x),1-\exp(-0.01(50-x))\right),\\
&\quad\times \min_{x\geq 0} \min(1,2-\exp(0.01\cdot x)-\exp(-0.01(50-x)))\}\\
&= 1-\exp(-0.01\cdot 50) = 0.39.
\end{aligned}
$$

Hence $\underline{Q}^{*}(50) = 0.61$ *and* $\overline{Q}^{*}(50) = 1$.

Probabilities on nested intervals

Consider a case with the following partial information about probabilities of failures:

$$
\underline{p}_{ij} \leq \Pr\{\underline{\alpha}_{ij} \leq X_{ij} \leq \overline{\alpha}_{ij}\} \leq \overline{p}_{ij},\ i=1,...,n,\ j=1,...,m_i, \tag{11}
$$

where

$$
[\underline{\alpha}_{i1},\overline{\alpha}_{i1}] \subset [\underline{\alpha}_{i2},\overline{\alpha}_{i2}] \subset ... \subset [\underline{\alpha}_{im_i},\overline{\alpha}_{im_i}],\ i=1,...,n. \tag{12}
$$

In other words, there are the nested intervals $[\underline{\alpha}_{ij},\overline{\alpha}_{ij}]$ with the interval probabilities $[\underline{p}_{ij},\overline{p}_{ij}]$ that a failure of the i-th component is inside these intervals, respectively. Here we have to note the additional condition $\overline{\alpha}_{i(m_i+1)} \to \infty$.

Introduce the following notation:

$$
V = \left\{(v_1,...,v_n) : \sum_{i=1}^{n} \overline{\alpha}_{v_i} \leq t\right\},
$$
$$
W = \left\{(w_1,...,w_n) : \sum_{i=1}^{n} \underline{\alpha}_{w_i} \geq t\right\}.
$$

Independent components

Proposition 3 *If the system components are statistically independent and governed by probabilities in the form* $\Pr\{\underline{\alpha}_{ij} \leq X_{ij} \leq \overline{\alpha}_{ij}\} = p_{ij}$ *and* $[\underline{\alpha}_{i1},\overline{\alpha}_{i1}] \subseteq ... \subseteq [\underline{\alpha}_{im_i},\overline{\alpha}_{im_i}]$, $i=1,...,n$, *then lower and upper bounds for the unreliability of a cold standby system at time* t *are computed as follows:*

$$
\underline{R}(t) = \sum_{V}\prod_{i=1}^{n}(p_{iv_i}-p_{i(v_i-1)}),\ \overline{R}(t) = 1-\sum_{W}\prod_{i=1}^{n}(p_{iw_i}-p_{i(w_i-1)}).
$$

Lack of knowledge about the independence of components

Proposition 4 *If the information about the cold standby system components is given as*

$$
\underline{p}_{ij} \leq \Pr\{\underline{\alpha}_{ij} \leq X_{ij} \leq \overline{\alpha}_{ij}\} \leq \overline{p}_{ij},\ i=1,...,n,\ j=1,...,m_i,
$$

then there hold

$$\underline{R}^*(t) = \max_V \max\left\{\sum_{i=1}^n \underline{p}_{iv_i} - (n-1), 0\right\},$$
$$\overline{R}^*(t) = 1 - \max_W \max\left\{\sum_{i=1}^n \underline{p}_{iw_i} - (n-1), 0\right\}.$$

Corollary 3 *If the information about a cold standby system is given as*

$$\underline{p}_i \leq \Pr\{\underline{\alpha} \leq X_i \leq \overline{\alpha}\} \leq \overline{p}_i,\ i = 1, ..., n,$$

then there hold for $t = \underline{\alpha}$

$$\underline{R}^*(t) = 0,\ \overline{R}^*(t) = 1 - \max\left\{\sum_{i=1}^n \underline{p}_i - (n-1), 0\right\}$$

and for $t = \overline{\alpha}$

$$\underline{R}^*(t) = 0,\ \overline{R}^*(t) = \begin{cases} 1, & t > n\underline{\alpha} \\ 1 - \max\left\{\sum_{i=1}^n \underline{p}_i - (n-1), 0\right\}, & t \leq n\underline{\alpha} \end{cases}.$$

It can be seen that the lower and upper bounds for the cold standby system unreliability depend only on lower probabilities of nested intervals. This implies that knowledge of upper probabilities does not give any useful information. Moreover, according to (Walley 1996), the initial information can be considered as the possibility and necessity measures (Dubois & Prade 1988). Indeed, according to (Dubois & Prade 1992), an upper probability induced by a set of lower bounds $\{P(A_i) \geq p_i, i = 1, ..., n\}$ is a possibility measure if the set $\{A_1, ..., A_n\}$ is nested, i.e., $A_1 \subset A_2 \subset ... \subset A_n$. Denote

$$\pi_i(\underline{\alpha}_j) = \pi_i(\overline{\alpha}_j) = 1 - \underline{p}_{ij},\ i = 1, ..., n,\ j = 1, ..., m_i.$$

Then the times to failure X_i of components can be regarded as fuzzy variables with the possibility distribution functions $\pi_i(\underline{\alpha}_j) = \pi_i(\overline{\alpha}_j)$, $i = 1, ..., n$, $j = 1, ..., m_i$.

Let us prove that the interval-valued system reliability unreliability by such initial data can be also considered as the possibility and necessity measures of failure before time t.

Proposition 5 *If initial information is represented as a set of probabilities defined on nested intervals, then either* $\underline{R}(t) = 0$ *or* $\overline{R}(t) = 1$.

It follows from Proposition 5 the definition of the possibility measures given in (Walley 1996) that if initial information is represented as a set of probabilities defined on nested intervals, then $\underline{R}(t)$ and $\overline{R}(t)$ can be regarded as the possibility and necessity measures, respectively. Then the possibility distribution function of the system time to failure can be obtained as follows (see Fig.3):

$$\pi_S(t) = \begin{cases} \overline{R}(t), & t \leq t_0 \\ 1, & t_0 \leq t \leq t_1 \\ 1 - \underline{R}(t), & t \geq t_1 \end{cases},$$

where $t_0 = \min\{t : \overline{R}(t) = 1\}$, or $t_1 = \max\{t : \underline{R}(t) = 0\}$.

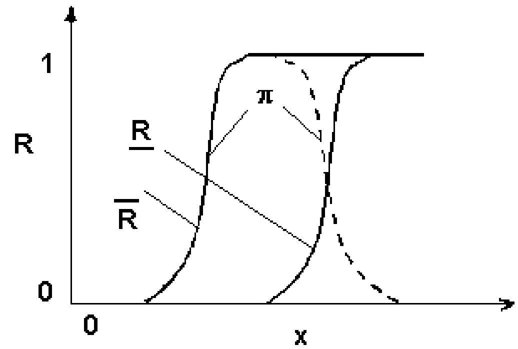

Figure 3: Lower and upper probability distributions and the possibility distribution function

The above reasoning allows us to obtain the reliability measure of a cold standby system by fuzzy initial data as a function Φ such that

$$\pi_S(t) = \Phi(\underline{p}_{ij},\ i = 1, ..., n,\ j = 1, ..., m_i.).$$

For example, there holds for the case of the lack of independence

$$\pi_S(t) = \begin{cases} 1 - \max_W \max\left\{\sum_{i=1}^n (1 - \pi_i(\underline{\alpha}_{w_i})) - (n-1), 0\right\}, & t \le t_0 \\ 1 - \max_V \max\left\{\sum_{i=1}^n (1 - \pi_i(\overline{\alpha}_{v_i})) - (n-1), 0\right\}, & t \ge t_0 \end{cases}$$

$$= \begin{cases} \min_W \min\left\{\sum_{i=1}^n \pi_i(\underline{\alpha}_{w_i}), 1\right\}, & t \le t_0 \\ \min_V \min\left\{\sum_{i=1}^n \pi_i(\overline{\alpha}_{v_i}), 1\right\}, & t \ge t_0 \end{cases}.$$

Example 3 *Let us consider a cold-standby system consisting of two identical components. Suppose that an expert provides the following judgements about reliability of components: 95% of all failures are between 10 and 300 days; 50% of all failures are between 30 and 200 days; 5% of all failures are between 70 and 100 days; The assessments of the expert can be represented as follows:*

$$\Pr(10 \le X \le 300) = 0.95,$$
$$\Pr(30 \le X \le 200) = 0.5,$$
$$\Pr(70 \le X \le 100) = 0.05.$$

Let us find bounds for the reliability of a system at time 50 *days, i.e., we have to find* $\overline{Q}(50) = 1 - \underline{R}(50)$ *and* $\underline{Q}(50) = 1 - \overline{R}(50)$. *By using notation introduced in this section, we can write*

$$\underline{\alpha}_{11} = \underline{\alpha}_{21} = 70,\ \underline{\alpha}_{12} = \underline{\alpha}_{22} = 30,\ \ \underline{\alpha}_{13} = \underline{\alpha}_{23} = 10,$$
$$\overline{\alpha}_{11} = \overline{\alpha}_{21} = 100,\ \overline{\alpha}_{12} = \overline{\alpha}_{22} = 200,\ \ \overline{\alpha}_{13} = \overline{\alpha}_{23} = 300,$$
$$p_{11} = p_{21} = 0.05,\ p_{12} = p_{22} = 0.5,\ \ p_{13} = p_{23} = 0.95,$$
$$p_{10} = p_{20} = 0.$$

Let us construct the sets $V = \{(v_1, v_2)\}$ *and* $W = \{(w_1, w_2)\}$*:*

$$V = \{\varnothing\},$$
$$W = \{(1,3), (3,1), (1,1), (1,2), (2,1), (2,2)\}.$$

1. *Components are independent. By using Proposition 3, we find* $\underline{R}(50)$ *and* $\overline{R}(50)$ *as follows:*

$$\begin{aligned}
\underline{R}(50) &= 0, \\
\overline{R}(50) &= 1 - (p_{11} - p_{10})(p_{23} - p_{22}) - (p_{13} - p_{12})(p_{21} - p_{20}) \\
&\quad - (p_{11} - p_{10})(p_{21} - p_{20}) - (p_{11} - p_{10})(p_{22} - p_{21}) \\
&\quad - (p_{12} - p_{11})(p_{21} - p_{20}) - (p_{12} - p_{11})(p_{22} - p_{21}) \\
&\quad = 1 - 0.295 = 0.705.
\end{aligned}$$

   Hence $\underline{Q}(50) = 0.295$ *and* $\overline{Q}(50) = 1$.

2. *Lack of knowledge about independence of components. By using Proposition 4, we can find*

$$\begin{aligned}
\underline{R}^*(50) &= 0, \\
\overline{R}^*(50) &= 1 - \max\{\max(p_{11} + p_{23} - 1, 0), \max(p_{13} + p_{21} - 1, 0), \\
&\times \max(p_{11} + p_{21} - 1, 0), \max(p_{11} + p_{22} - 1, 0), \\
&\times \max(p_{12} + p_{21} - 1, 0), \max(p_{12} + p_{22} - 1, 0)\} \\
&= 1 - 0 = 1.
\end{aligned}$$

   Hence $\underline{Q}^*(50) = 0$ *and* $\overline{Q}^*(50) = 1$. *These results illustrate that it is impossible to forecast the system reliability by such non-informative initial data and by the lack of knowledge about independence of components.*

Practical relevance of results

One of the main aims of using the cold standby redundancy in systems is achievement of a required level of the system reliability. A number of redundant components is determined by the required level of reliability and by the component reliability. If there exists complete information about the system reliability behavior (precise probability distributions of the component time to failure are known and components are independent), then the problem of computing the optimal number of redundant components can be always solved at least theoretically. However, information about reliability of components may be restricted by judgements of experts, especially, if the analyzed system contains new components and there is no complete statistical data. In this case, we have only some partial information about reliability of components and the problem of the optimal reserve becomes more complex. Of course, we can assume some typical probability distribution of the component time to failure and to find the number of redundant components by means of well-known methods. But Example 2 shows how resulting reliability measures may differ in this case ($\underline{Q}(50) = \overline{Q}(50) = 0.91$ by assuming the exponential distribution of times to failure, $\underline{Q}(50) = 0.7$ and $\overline{Q}(50) = 0.9975$ by using only three points of the exponential distribution of times to failure). This difference may lead to errors in determining the optimal system reserve and even to catastrophic consequences. Therefore, the obtained analytical expressions for reliability of cold standby systems are vitally important because the numerical solution of optimization problems like (1)-(3) and (4)-(6) is a very complex task.

The second question is how to use the obtained interval reliability measures. It is worth noticing that requirements to the system reliability are usually given in the form of some precise values. This leads to a problem of comparison of imprecise and precise reliability measures. This procedure depends on a decision maker and the system purposes (consequences of failures). In any case, a resulting decision can fall into the range from pessimistic to optimistic. If consequences of the system failure are catastrophic (transport systems, nuclear power plants, weapon), then lower bounds (pessimistic decision) for the system reliability have to be determinative and are compared with the required level of the system reliability. If the system failure does not imply major consequences, then upper bounds (optimistic decision) can be used. Generally, the decision maker may use a caution parameter η for comparison of imprecise and precise reliability measures on the basis of his (her) own experience, various conditions of the system functioning and so on. In this case, the precise value of the system reliability is determined as the linear combination $\eta \underline{Q}(t) + (1-\eta)\overline{Q}(t)$. At that, if $\eta = 0$, then we get the optimistic result. If $\eta = 1$, then the pessimistic view is determinative.

Conclusion

It has been shown that the reliability assessments of cold standby systems depend on available information about reliability behavior of components. The results also differ with respect to the judgement of independency of components. It is clear, the less judgements are used, the assessment of reliability is more imprecise, i.e., the imprecision of results reflects insufficiency of available information. It should be noted that the systems have been analyzed without information about the certain probability distributions of the component times to failure. And this makes the reliability calculation to be more realistic. Moreover, the obtained results have the strong mathematical sense and can be widely used in practice.

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Appendix

Proof of Proposition 1: Let us consider a system consisting of two components for simplicity. It was proven in (Utkin & Kozine 2001) that solutions to optimization problems (8)-(10) exist on degenerate distributions. Referring to this property, the following optimization problems, equivalent to (8)-(10), can be stated:

$$\underline{R}(t)(\overline{R}(t)) = \min\,(\max) \sum_{k=1}^{m_1+1} \sum_{j=1}^{m_2+1} I_{[0,t]}(x_{1k} + x_{2j}) c_k d_j, \tag{13}$$

subject to

$$\sum_{k=1}^{m_1+1} c_k = 1, \ \sum_{i=1}^{m_2+1} d_i = 1, \tag{14}$$

$$\sum_{i=1}^{m_1+1} I_{[0,\alpha_{1k}]}(x_{1i})c_i = p_{1k},\ \sum_{i=1}^{m_2+1} I_{[0,\alpha_{2j}]}(x_{2i})d_i = p_{2k},\ k \leq m_1,\ j \leq m_2. \tag{15}$$

Here the minimum and maximum are taken over a set of variables x_{1i}, x_{2j}, c_i, $d_j \in \mathbb{R}_+$, $i \leq m_1$, $j \leq m_2$, subject to constraints (14)-(15). Assume that $x_{i1} \leq ... \leq x_{i(m_i+1)}$ are the values delivering $\min$ and $\max$ to objective function (13). Suppose that there are two optimal values of x_{ij} and x_{ik} such that $x_{ij} \in [\alpha_{i(k-1)}, \alpha_{ik}]$ and $x_{ik} \in [\alpha_{i(k-1)}, \alpha_{ik}]$. If $i = 1$ and $j < k$, then it follows from (15) that $c_1 + ... + c_j = p_{1k}$ and $c_1 + ... + c_{j+1} = p_{1k}$, which is a contradiction. The same contradiction is obtained if $j > k$. Similarly, we arrive at contradictions for an arbitrary number of values x_{1k} belonging to the same interval and for $i = 2$. This implies that $x_{ik} \in [\alpha_{i(k-1)}, \alpha_{ik}]$. It follows from these conditions and from (15) that $c_k = p_{1k} - p_{1(k-1)}$, $d_j = p_{2j} - p_{2(j-1)}$, $k \leq m_1$, $j \leq m_2$. The inequality $x_{1k} + x_{2j} \leq t$ is valid for any $x_{1k} \in [\alpha_{1(k-1)}, \alpha_{1k}]$ and $x_{2j} \in [\alpha_{2(j-1)}, \alpha_{2j}]$, and $I_{[0,t]}(x_{1k} + x_{2j}) = 1$ if there holds $\alpha_{1k} + \alpha_{2j} \leq t$. This implies that

$$\begin{aligned}\underline{R}(t) &= \min \sum_{k=1}^{m_1+1}\sum_{j=1}^{m_2+1} I_{[0,t]}(x_{1k} + x_{2j})c_k d_j = \sum_{(k,j)\in V} c_k d_j \\ &= \sum_{(k,j)\in V} (p_{1k} - p_{1(k-1)})(p_{2j} - p_{2(j-1)}).\end{aligned}$$

Similarly, we can find $\overline{R}(t)$. The generalization on the case of n components is obvious. ■

Proof of Corollary 1: Let us consider a system consisting of two components for simplicity. It follows from Proposition 1 that

$$\underline{R}(t) = \lim_{\Delta x \to 0} \sum_{V} (p_1(x) - p_1(x - \Delta x))(p_2(z) - p_2(z - \Delta z)).$$

Here $p_i(x) = p_{ik}$, $p_i(x - \Delta x) = p_{i(k-1)}$, the set V contains an infinite number of real numbers x and z such that there holds $x + z \leq t$. Hence

$$\begin{aligned}\underline{R}(t) &= \lim_{\Delta x \to 0, \Delta z \to 0} \sum_{y \leq t}\sum_{x+z=y} \frac{(p_1(x) - p_1(x - \Delta x))}{\Delta x}\frac{(p_2(z) - p_2(z - \Delta z))}{\Delta z}\Delta x \Delta z \\ &= \int_0^t \int_0^y f_1(x) f_2(y - x)\mathrm{d}x\mathrm{d}y = \int_0^t f_1 * f_2(y)\mathrm{d}y.\end{aligned}$$

The upper bound $\overline{R}(t)$ can be obtained in the same way. ■

Proof of Proposition 2: Let us consider a system consisting of two components for simplicity. Introduce notation: D is the event $\{X_1 + X_2 \leq t\}$, A_i is the event $\{X_1 \in [0, \alpha_{1i}]\}$ and A_i^c is the set complement to A_i, B_i is the event $\{X_2 \in [0, \alpha_{2i}]\}$, and $A_i B_k$ is a subset of the universal set $A_{m_1+1} \times B_{m_2+1}$. By using the proof of Proposition 1 and expressions for computing lower and upper probabilities of an event on the basis of available probabilities of some events (Kuznetsov 1991), we can write

$$\underline{R}^*(t) = \max\left\{ \max_{i,j:D \supset A_i B_j} \underline{P}(A_i B_j), 1 - \min_{i,j:D^c \subset A_i B_j} \overline{P}(A_i B_j) \right\}.$$

The first condition $D \supset A_i B_j$ is valid for all $A_i B_j$ such that $(i,j) \in V$. The second condition $D^c \subset A_i B_j$ is valid only for $A_i = [0,\infty)$ and $B_j = [0,\infty)$. This implies that

$$\begin{aligned}\underline{R}^*(t) &= \max\left\{\max_{(i,j)\in V}\underline{P}(A_iB_j), 1-\overline{P}(A_{m_1+1}B_{m_2+1})\right\} \\ &= \max_{(i,j)\in V}\max\left\{(\underline{p}_{1i}+\underline{p}_{2j}-1), 0\right\}.\end{aligned}$$

The upper bound can be found as

$$\overline{R}^*(t) = \min\left\{\min_{i,j:D\subset A_iB_j}\overline{P}(A_iB_j), 1-\max_{i,j:D^c\supset A_iB_j}\underline{P}(A_iB_j)\right\}.$$

The first condition $D\subset A_iB_j$ is valid if $\alpha_{1i}\geq t$ and $\alpha_{2j}\geq t$, i.e., $(i,j)\in S$. The second condition $D^c\supset A_iB_j$ is valid if $(i+1,j+1)\in W$ and events $A_i^cB_j^c$ are taken into account. This implies that

$$\overline{R}^*(t) = \min\left\{\min_{(i,j)\in S}\min\left(\overline{p}_{1i},\overline{p}_{2j}\right), \min_{(i,j)\in W}\min\left(\overline{p}_{1(i-1)}+\overline{p}_{2(j-1)}, 1\right)\right\}.$$

The generalization on the case of n components is obvious. ■

Proof of Corollary 2: The formulas can be obtained directly from Proposition 2 and the proof of Corollary 1. ■

Proof of Proposition 3: The proof is similar to the proof of Proposition 1. Here if $\overline{\alpha}_{1j}+\overline{\alpha}_{2k}\leq t$, then the sum of any points in intervals $[\underline{\alpha}_{1j},\overline{\alpha}_{1j}]\backslash[\underline{\alpha}_{1(j-1)},\overline{\alpha}_{1(j-1)}]$ and $[\underline{\alpha}_{2k},\overline{\alpha}_{2k}]\backslash[\underline{\alpha}_{2(k-1)},\overline{\alpha}_{2(k-1)}]$ is less than t. Hence, we obtain $\underline{R}$. The upper bound can be obtained similarly. ■

Proof of Proposition 4: Similarly to the proof of Proposition 2. ■

Proof of Corollary 3: If $t=\underline{\alpha}\leq\overline{\alpha}$, then $V=\{\emptyset\}$, $W=\{(1,1,...,1)\}$. If $t=\overline{\alpha}$, then $V=\{\emptyset\}$, $W=\{(1,1,...,1)\}$ by $t\leq n\underline{\alpha}$ and $W=\{\emptyset\}$ by $t>n\underline{\alpha}$. ■

Proof of Proposition 5: It follows from the definition of the sets V and W that if the set V is non-empty and $\underline{R}(t)\geq 0$, i.e., there exists at least one vector $(v_1,...,v_n)$ such that $\sum_{i=1}^n\overline{\alpha}_{v_i}\leq t$, then there holds

$$\sum_{i=1}^n\underline{\alpha}_{w_i}\leq\sum_{i=1}^n\overline{\alpha}_{v_1}\leq\sum_{i=1}^n\overline{\alpha}_{v_i}\leq t.$$

This implies that the set W is empty and $\overline{R}(t)=1$. The equality $\underline{R}(t)=0$ is similarly proved. ■