Interval reliability of typical systems with partially known probabilities

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Abstract

Most probabilistic methods of the system reliability analysis assume that the component lifetime distributions are known precisely and the system components are independent. The paper considers reliability analysis of typical systems when the above assumptions are violated and there is only some partial information about the component lifetime distributions. The analysis is based on considering sets of possible probability distributions consistent with available information. All resulting expressions are obtained in the explicit form. The impact of incompleteness of initial data on the resulting system reliability measures is illustrated by a numerical example.

Keywords: reliability, uncertainty modelling, imprecise probabilities, incomplete information, series and parallel systems, lower and upper bounds.

1 Introduction

Most probabilistic methods of reliability analysis of engineering systems assume that all probabilities are precise, that is, that every probability involved is perfectly determinable. Moreover, it is usually assumed that there exists some complete probabilistic information about the system and component behavior. The completeness of the probabilistic information means that two conditions must be fulfilled:

1. all probabilities or probability distributions are known or perfectly determinable;
2. the system components are independent, i.e., all random variables, describing the component reliability behavior, are independent.

The precise system reliability measures can be always (at least theoretically) computed if both conditions are satisfied (it is assumed here that the system structure is defined precisely and there exists a function linking the system time to failure and times to failure of components). If at least one condition is violated, then only the interval reliability measures can be obtained. In reality, it is difficult to expect that the first condition is fulfilled. If a system is new or exists only as a project, then there are not sufficient statistical data in the most cases. Even if such data exist, we do not always observe their stability from the statistical point of view. This implies that only some partial information about the system components is known, for example, the mean time to failure or bounds for the probability of failure at a time. Of course, one can always assume that the time to failure has a certain distribution, for example, exponential or normal. However, how to belief to obtained results of reliability analysis if our assumption is based only on our or expert's experience. A number of methods are devoted to computing the reliability bounds under different assumptions about initial incomplete data [1, 2, 3, 7]. The main idea in this paper is to model the violation of the first condition by sets of possible probability distributions of times to failure consistent with the available information and to compute the lower and upper reliability probabilistic measures in accordance with these sets.

It is difficult to expect that components of many systems are independent. Let us consider two programs functioning in parallel. If these programs were developed by means of the same programming language, then possible errors in a language library of typical functions produce dependent faults in both programs. Several experimental studies show [4] that the assumption of independence of failures between independently developed programs does not hold. Moreover, the main difficulty here is that the degree of dependency is unknown. The same examples can be presented for other applications. This implies that the second condition for complete information is also violated and it is impossible to obtain some precise reliability measures for a system. This fact was also pointed out by Barlow and Proschan in [2]. The lack of information about independence is modelled in this paper by considering sets of joint lifetime distributions, but not products of marginal ones.

Reliability analysis of typical systems (series and parallel) under partial information about probabilities of times to failure of the system components and under the lack of information about independence of the system components is studied in this paper. All expressions are obtained in the explicit form. I consider the simplest typical systems in order to investigate in a more clear form the impact of incompleteness of available information on the resulting system reliability measures. It should be noted that the proposed methods for computing reliability differ from methods of the well-known interval computation. Here I deal with sets of probability distributions, but not with sets of possible values of reliability measures in intervals. As a result, I obtain the best bounds for the system reliability under given information.

In Section 2, a general approach for reliability analysis of arbitrary systems under various types of the initial probabilistic information is considered. Section 3 is devoted to computing the interval reliability measures of series and parallel systems under condition that there exist only some points of lower and upper probability distributions of the component times to failure. The condition of the lack of information about independence of components is also studied in this section. In Section 4, it is assumed that the interval probabilities of failures are defined on nested intervals. Properties of the system duality are focused in Section 5. In Section 6, the proposed approach is illustrated by a numerical example.

2 Reliability of systems under partial information

Consider a system consisting of n components. Let $\varphi_{ij}(X_i)$ be a function of the random time to failure X_i of the i-th component. For methods that are presented next there is no difference between 'failure times' (for technical units in a reliability study) or 'lifetimes' (for survival analyses). Therefore, we will use both terms in this paper. According to [2], the system lifetime can be uniquely determined by the component lifetimes. Denote $X = (X_1, ..., X_n)$ and $\mathbf{X} = (x_1, ..., x_n)$. Then there exists a function $g(X)$ of the component lifetimes characterizing the system reliability behavior. Suppose that partial information is represented as a set of the lower and upper expectations $\underline{\mathbb{E}}\varphi_{ij}$ and $\overline{\mathbb{E}}\varphi_{ij}$, $i = 1, ..., n$, $j = 1, ..., m_i$, of the functions φ_{ij}. Here m_i is the number of judgements that are related to the i-th component. For example, an interval-valued probability of failure in the interval $[a, b]$ can be represented as expectations of the indicator function $I_{[a,b]}(X)$ such that $I_{[a,b]}(X) = 1$ if $X \in [a, b]$, and $I_{[a,b]}(X) = 0$ if $X \notin [a, b]$.

A system reliability measure can be regarded as the expectation $\mathbb{E}g$ of the function g

$$\mathbb{E}g = \int_{\mathbb{R}_+^n} g(\mathbf{X})\rho(\mathbf{X})\mathrm{d}\mathbf{X},$$

where ρ is a probability density function of the system time to failure.

However, we do not know the density ρ because our initial information is restricted only by the lower and upper expectations $\underline{\mathbb{E}}\varphi_{ij}$ and $\overline{\mathbb{E}}\varphi_{ij}$, $i = 1, ..., n$, $j = 1, ..., m_i$, and there is no information about distributions of the component time to failure. At the same time, the available lower and upper expectations form a set $\mathcal{P}$ of possible densities that are consistent with these expectations. This means that we can find only the largest and smallest possible values of $\mathbb{E}g$ for all densities from the set $\mathcal{P}$. It can be carried out by solving the following optimization problems [5, 6, 9, 12]:

$$\underline{\mathbb{E}}g = \min_{\mathcal{P}} \int_{\mathbb{R}_+^n} g(\mathbf{X})\rho(\mathbf{X})\mathrm{d}\mathbf{X},\ \overline{\mathbb{E}}g = \max_{\mathcal{P}} \int_{\mathbb{R}_+^n} g(\mathbf{X})\rho(\mathbf{X})\mathrm{d}\mathbf{X}, \tag{1}$$

subject to

$$\int_{\mathbb{R}_+^n} \rho(\mathbf{X})\mathrm{d}\mathbf{X} = 1,\ \rho(\mathbf{X}) \geq 0,$$

$$\underline{\mathbb{E}}\varphi_{ij} \leq \int_{\mathbb{R}_+^n} \varphi_{ij}(x_i)\rho(\mathbf{X})\mathrm{d}\mathbf{X} \leq \overline{\mathbb{E}}\varphi_{ij},\ i \leq n,\ j \leq m_i. \tag{2}$$

Here the minimum and maximum are taken over the set $\mathcal{P}$ of all possible n-dimensional density functions $\{\rho(\mathbf{X})\}$ satisfying conditions (2). In other words, solutions to the problems are defined on the set $\mathcal{P}$ of possible densities that are consistent with the partial information expressed in the form of constraints (2).

It should be noted that only joint densities are used in (1)-(2) because, in a general case, we may not be aware whether the variables $X_1, ..., X_n$ are dependent or not. If it is known that components are independent, then $\rho(\mathbf{X}) = \rho(x_1) \cdots \rho(x_n)$. In this case, the set $\mathcal{P}$ is reduced and consists only of the densities that can be represented as a product. The optimization problems for computing new bounds for the expectation of g are of the form:

$$\underline{\mathbb{E}}g(\overline{\mathbb{E}}g) = \min_{\mathcal{P}} \left(\max_{\mathcal{P}}\right) \int_{\mathbb{R}_+^n} g(\mathbf{X})\rho_1(x_1) \cdots \rho_n(x_n)\mathrm{d}\mathbf{X}, \tag{3}$$

subject to

$$\rho_i(x_i) \geq 0,\ \int_0^\infty \rho_i(x_i)\mathrm{d}x_i = 1,$$

$$\underline{\mathbb{E}}\varphi_{ij} \leq \int_0^\infty \varphi_{ij}(x_i)\rho_i(x_i)\mathrm{d}x_i \leq \overline{\mathbb{E}}\varphi_{ij},\ i \leq n,\ j \leq m_i. \tag{4}$$

In this case, the problems are non-linear and this fact makes them more difficult to be solved. However, some reliability problems can be easily solved by using this form. For example, the system lower and upper mean times to failure for various systems by the known component mean times to failure have been obtained in [5].

If the function g is the indicator function of an event A, i.e., $g(X) = I_A(X)$, then $\underline{\mathbb{E}}g$ and $\overline{\mathbb{E}}g$ are the lower and upper probabilities of A. We will denote these probabilities $\underline{P}(A)$ and $\overline{P}(A)$. According to [6, 11, 13], optimization problems (1)-(2) and (3)-(4) produce coherent results in this case. This means that $\underline{P}(A) = 1 - \overline{P}(A^c)$ and $\overline{P}(A) = 1 - \underline{P}(A^c)$. Here A^c is the complement to A.

Reliability analysis for some important special cases of initial information is considered in this paper. At that, the calculated reliability measure is the probability $R(t)$ of failure before time t, i.e.,

$$R(t) = \Pr\{g(X) \leq t\} = \mathbb{E}I_{[0,t]}(g).$$

This measure is called the unreliability. The reliability

$$Q(t) = \Pr\{g(X) \geq t\} = \mathbb{E}I_{[t,\infty)}(g)$$

can be found from the condition $Q(t) = 1 - R(t)$. If the system reliability measures are interval-valued, then $\overline{Q}(t) = 1 - \underline{R}(t)$, and $\underline{Q}(t) = 1 - \overline{R}(t)$.

3 Partially known probability distributions

Assume that initial information about the times to failure X_i of the system components is given in the following form:

$$\underline{p}_{ij} \leq \Pr\{X_i \leq \alpha_{ij}\} \leq \overline{p}_{ij},\ i = 1,...,n,\ j = 1,...,m_i. \tag{5}$$

Here it is assumed that

$$\alpha_{i1} \leq \alpha_{i2} \leq ... \leq \alpha_{im_i}. \tag{6}$$

This assumption is obvious because $\underline{p}_{ij}$, $\overline{p}_{ij}$, $j = 1,...,m_i$, are values of interval probability distributions. In other words, only m_i points of the probability distribution of X_i, $i = 1,...,n$, are known with some accuracy. If there is only partial information about probability distributions in the form of (5)-(6), then the condition of consistency of probabilities is [6]

$$\underline{p}_{ik} \leq \overline{p}_{ij},\ \forall k \leq j. \tag{7}$$

Below we assume that all available probabilities are consistent because optimization problems (1)-(2) and (3)-(4) do not have any solution in the case of inconsistency since their feasible regions are empty.

3.1 Independent components

Suppose the variables X_i, $i = 1,...,n$, are independent. Then (3)-(4) can be rewritten as

$$\underline{R}(t)(\overline{R}(t)) = \min_{\mathcal{P}}(\max_{\mathcal{P}}) \int_{\mathbb{R}^n_+} I_{[0,t]}(g(\mathbf{X}))\rho_1(x_1)\cdots\rho_n(x_n)\mathrm{d}\mathbf{X}, \tag{8}$$

subject to

$$\int_{\mathbb{R}_+} \rho_i(x)\mathrm{d}x = 1,\ \rho_i(x) \geq 0,\ i = 1,...,n$$

$$\underline{p}_{ij} \leq \int_{\mathbb{R}_+} I_{[0,\alpha_{ij}]}(x)\rho_i(x)\mathrm{d}x \leq \overline{p}_{ij},\ i = 1,...,n,\ j = 1,...,m_i. \tag{9}$$

Without loss of generality, it is assumed $\underline{p}_{i0} = \overline{p}_{i0} = 0$, $\underline{p}_{i(m_i+1)} = \overline{p}_{i(m_i+1)} = 1$, $\alpha_{i0} = 0$, $\alpha_{i(m_i+1)} = T_i$. In particular, one can assume $T_i \to \infty$.

Let v_i be the minimal number j such that $\alpha_{ij} \geq t$, i.e.,

$$v_i = \min\{j : \alpha_{ij} \geq t\},$$

and let w_i be the maximal number j such that $\alpha_{ij} \leq t$, i.e.,

$$w_i = \max\{j : \alpha_{ij} \leq t\}.$$

If components are identical, then the index i is omitted.

3.1.1 Series systems

Definition 1 *A system is called series if its time to failure $g(X)$ is given by $g(X) = \min_{i=1,...,n} X_i$.*

Theorem 1 *If the system components are statistically independent and governed by consistent partially known probability distributions in the form of (5)-(6), then the lower and upper bounds for the unreliability of the series system at time t are computed as follows:*

$$\underline{R}(t) = 1 - \prod_{i=1}^{n}(1 - \underline{p}_{iw_i}),\ \overline{R}(t) = 1 - \prod_{i=1}^{n}(1 - \overline{p}_{iv_i}). \tag{10}$$

Proof. Let us consider a system consisting of two components for simplicity. First, we assume that $\underline{p}_{ij} = \overline{p}_{ij} = p_{ij}$, for all i and j. It was proven in [10] that solutions to optimization problems (8)-(9) exist on degenerate distributions. Referring to this property, the following optimization problems, equivalent to (8)-(9), can be stated:

$$\underline{R}(\overline{R}) = \min\,(\max) \sum_{k=1}^{m_1+1} \sum_{j=1}^{m_2+1} I_{[0,t]}(\min(x_{1k}, x_{2j}))c_k d_j, \tag{11}$$

subject to

$$\sum_{k=1}^{m_1+1} c_k = 1,\ \sum_{i=1}^{m_2+1} d_i = 1, \tag{12}$$

$$\sum_{i=1}^{m_1+1} I_{[0,\alpha_{1k}]}(x_{1i})c_i = p_{1k},\ \sum_{i=1}^{m_2+1} I_{[0,\alpha_{2j}]}(x_{2i})d_i = p_{2k},\ k = 1, ..., m_1,\ j = 1, ..., m_2. \tag{13}$$

Here the minimum and maximum are taken over the set of the variables $x_{1i},\ x_{2j},\ c_i,\ d_j \in \mathbb{R}_+,\ i = 1, ..., m_1$, $j = 1, ..., m_2$, subject to constraints (12)-(13).

Assume that

$$x_{11} \leq x_{12} \leq ... \leq x_{1(m_1+1)},\ y_{11} \leq y_{12} \leq ... \leq y_{1(m_2+1)}$$

are the values delivering $\min$ and $\max$ to objective function (11). Let us prove that the values x_{1k} and x_{2k} delivering the optima meet the following conditions: $x_{1k} \in [\alpha_{1(k-1)}, \alpha_{1k}]$ and $y_{2k} \in [\alpha_{2(k-1)}, \alpha_{2k}]$ for all possible k. Suppose that there are two optimal values of x_{1j} and x_{1k} such that $x_{1j} \in [\alpha_{1(k-1)}, \alpha_{1k}]$ and $x_{1k} \in [\alpha_{1(k-1)}, \alpha_{1k}]$. If $j < k$, then it follows from (13) that

$$c_1 + ... + c_j = p_{1k},\ c_1 + ... + c_j + c_{j+1} = p_{1k},$$

which is a contradiction. If $j > k$, then it follows from (13) that

$$c_1 + ... + c_k = p_{1k},\ c_1 + ... + c_k + c_{k+1} = p_{1k},$$

which is also a contradiction. Similarly, we arrive at contradictions for an arbitrary number of values x_{1k} belonging to the same interval. This implies that $x_{1k} \in [\alpha_{1(k-1)}, \alpha_{1k}]$. The proof of the condition $x_{2k} \in [\alpha_{2(k-1)}, \alpha_{2k}]$ can be obtained in a similar way.

It follows from these conditions ($x_{1k} \in [\alpha_{1(k-1)}, \alpha_{1k}]$ and $x_{2k} \in [\alpha_{2(k-1)}, \alpha_{2k}]$) and from (13) that

$$c_1 = p_{11},\ c_1 + c_2 = p_{12}, ..., \sum_{i=1}^{m_1} c_i = p_{1m_1},$$

$$d_1 = p_{21},\ d_1 + d_2 = p_{22}, ..., \sum_{i=1}^{m_2} d_i = p_{2m_2}.$$

Hence

$$c_k = p_{1k} - p_{1(k-1)},\ d_j = p_{2j} - p_{2(j-1)},\ k = 1, ..., m_1,\ j = 1, ..., m_2.$$

Then

$$\begin{aligned}\overline{R} &= \overline{P}\left(\min(X_1, X_2) \leq t\right) = 1 - \underline{P}\left(\min(X_1, X_2) \geq t\right) \\ &= 1 - \inf \sum_{k=1}^{m_1+1} \sum_{j=1}^{m_2+1} I_{[t,\infty)}(\min(x_{1k}, x_{2j}))c_k d_j. \qquad (14)\end{aligned}$$

Note that function (14) achieves its minimum if there hold $I_{[t,\infty)}(\min(x_{2j}, x_{1k})) = 0$ for all $k \leq m_1 + 1$ and $j \leq m_2 + 1$. However, there exist the values j and k such that $I_{[t,\infty)}(\min(x_{2j}, x_{1k})) = 1$ for some combinations of x_{2j} and x_{1k}. Then $I_{[t,\infty)}(\min(x_{2j}, x_{1k})) = 1$ for all x_{2j} and x_{1k} such that $j \geq v_1$ and $k \geq v_2$. Thus, it can be concluded

$$\overline{R} = 1 - \sum_{k=v_1+1}^{m_1+1} \sum_{j=v_2+1}^{m_2+1} c_k d_j = 1 - \sum_{k=v_1+1}^{m_1+1} \sum_{j=v_2+1}^{m_2+1} (p_{1k} - p_{1(k-1)})(p_{2j} - p_{2(j-1)}).$$

Taking into account $\sum_{j=v_1+1}^{m_1+1}(p_{1j} - p_{1(j-1)}) = 1 - p_{1v_1}$ and $\sum_{j=v_2+1}^{m_2+1}(p_{2j} - p_{2(j-1)}) = 1 - p_{2v_2}$, the last formula is reduced to

$$\overline{R} = 1 - (1 - p_{1v_1})(1 - p_{2v_2}).$$

Here $\overline{R}$ increases as p_{iv_i}, $i = 1, ..., n$, increase. This implies that

$$\overline{R} = \max_{\underline{p}_{ij} \leq p_{ij} \leq \overline{p}_{ij}} \{1 - (1 - p_{1v_1})(1 - p_{2v_2})\} = 1 - (1 - \overline{p}_{1v_1})(1 - \overline{p}_{2v_2}).$$

For computing the lower bound $\underline{R}$, one can write

$$\underline{R} = \underline{P}\left(\min(X_1, X_2) \leq t\right) = \inf \sum_{k=1}^{m_1+1} \sum_{j=1}^{m_2+1} I_{[0,t]}(\min(x_{1k}, x_{2j}))c_k d_j$$

$$= c_1(d_1 + ... + d_{m_2+1}) + ... + c_{w_1}(d_1 + ... + d_{m_2+1})$$

$$+ d_1(c_{w_{1+1}} + ... + c_{m_1+1}) + ... + d_{w_2}(c_{w_1+1} + ... + c_{m_1+1}) = 1 - (1 - p_{1w_1})(1 - p_{2w_2}).$$

Here $\underline{R}$ decreases as p_{iw_i}, $i = 1, ..., n$, decrease. This implies that

$$\underline{R} = \min_{\underline{p}_{ij} \leq p_{ij} \leq \overline{p}_{ij}} \{1 - (1 - p_{1w_1})(1 - p_{2w_2})\} = 1 - (1 - \underline{p}_{1w_1})(1 - \underline{p}_{2w_2}).$$

The generalization on the case of n components is obvious. ■

Corollary 1 *If the system components are statistically independent and the probability distributions* $F_i(t) = \Pr(X_i \leq t)$ *of their times to failure are known precisely, then*

$$\underline{R}(t) = \overline{R}(t) = 1 - \prod_{i=1}^{n}(1 - F_i(t)).$$

Proof. This follows from Theorem 1 and from the conditions $v_i = \min\{j : \alpha_{ij} \geq t\}$, $w_i = \max\{j : \alpha_{ij} \leq t\}$, $\alpha_{ij} = t$, $p_{iv_i} = p_{iw_i} = F_i(t)$. ■

Corollary 2 *If the system components are identical, then there hold*

$$\underline{R}(t) = 1 - (1 - \underline{p}_w)^n, \ \overline{R}(t) = 1 - (1 - \overline{p}_v)^n.$$

Proof. The proof is obvious. ■

Corollary 3 *The lower system unreliability is defined only by probabilities of the largest intervals* $[0, \alpha_{w_i}]$ *which are included in* $[0, t]$. *The upper system unreliability is defined only by probabilities of the smallest intervals* $[0, \alpha_{v_i}]$ *which include* $[0, t]$.

Proof. The proof is obvious from (10). ■

3.1.2 Parallel systems

Definition 2 *A system is called parallel if its time to failure* $g(X)$ *is given by* $g(X) = \max_{i=1,...,n} X_i$.

Theorem 2 *If the system components are statistically independent and governed by consistent partially known probability distributions in the form of (5)-(6), then the lower and upper bounds for the unreliability of the parallel system at time* t *are computed as follows:*

$$\underline{R}(t) = \prod_{i=1}^{n} \underline{p}_{iw_i}, \ \overline{R}(t) = \prod_{i=1}^{n} \overline{p}_{iv_i}.$$

Proof. Similarly to the proof of Theorem 1. ■

Corollary 4 *If the system components are statistically independent and the probability distributions* $F_i(t) = \Pr(X_i \leq t)$ *of their times to failure are known precisely, then*

$$\underline{R}(t) = \overline{R}(t) = \prod_{i=1}^{n} F_i(t).$$

Proof. Similarly to the proof of Corollary 1. ■

Corollaries 1 and 4 state that the obtained expressions coincide with conventional ones known in the reliability theory. This means that Theorems 1 and 2 generalize the conventional formulas for reliability to interval-valued probabilities.

Corollary 5 *If the system components are identical, then there hold*

$$\underline{R}(t) = \underline{p}_w{}^n, \ \overline{R}(t) = \overline{p}_v{}^n.$$

Proof. The proof is obvious. ■

Corollary 6 *The lower system unreliability is defined only by probabilities of the largest intervals* $[0, \alpha_{w_i}]$ *which are included in* $[0, t]$*. The upper system unreliability is defined only by probabilities of the smallest intervals* $[0, \alpha_{v_i}]$ *which include* $[0, t]$.

Proof. The proof is obvious from Theorem 2. ■

3.2 Lack of knowledge about independence

It was assumed in the previous sections that the system components are independent. Now we remove this additional assumption and suppose that there is no information about independence of components.

The asterisk notation in $\underline{R}^*$ and $\overline{R}^*$ will mean that the bounds for the unreliability are obtained based on the lack of information about independence of components.

3.2.1 Series systems

Theorem 3 *If the system components are not judged to be independent, then the lower and upper bounds for the unreliability at time* t *of the series system are computed as follows:*

$$\underline{R}^*(t) = \max_{i=1,\dots,n} \underline{p}_{iw_i}, \ \overline{R}^*(t) = \min\left(1, \sum_{i=1}^{n} \overline{p}_{iv_i}\right).$$

Proof. Let us consider a series system consisting of two components for simplicity. Let us introduce notation: D is the event $\{\min(X_1, X_2) \geq t\}$, A_i is the event $\{X_1 \in [0, \alpha_{1i}]\}$ and A_i^c is the set complement to A_i, B_i is the event $\{X_2 \in [0, \alpha_{2i}]\}$, and $A_i B_k$ is a subset of the universal set $A_{m_1+1} \times B_{m_2+1}$. By using the proof of Theorem 1 and expressions for computing lower and upper probabilities of an event on the basis of available probabilities of some events [6], we can write

$$\begin{aligned}
\overline{R}^* &= 1 - \underline{P}(D) = 1 - \max\left\{\max_{i,j:D\supset A_iB_j} \underline{P}(A_iB_j), 1 - \min_{i,j:D^c\subset A_iB_j} \overline{P}(A_iB_j)\right\} \\
&= 1 - \max\left\{\max_{i\geq v_1, j\geq v_2} \underline{P}(A_i^cB_j^c), 1 - \overline{P}(A_{m_1+1}B_{m_2+1})\right\} \\
&= 1 - \max\left\{\max_{i\geq v_1, j\geq v_2} \max\left(\underline{P}(A_i^c) + \underline{P}(B_j^c) - 1, 0\right), 0\right\} \\
&= 1 - \max\left\{\max_{i\geq v_1, j\geq v_2} \max\left(1 - \overline{p}_{1i} - \overline{p}_{2j}, 0\right)\right\} = \min\left(1, \overline{p}_{1v} + \overline{p}_{2w}\right).
\end{aligned}$$

$$\begin{aligned}
\underline{R}^* &= 1 - \overline{P}(D) = 1 - \min\left\{\min_{i,k:D\subset A_iB_k} \overline{P}(A_iB_k), 1 - \max_{i,k:D^c\supset A_iB_k} \underline{P}(A_iB_k)\right\} \\
&= 1 - \min\left\{\min_{i\leq w_1, k\leq w_2} \overline{P}(A_i^cB_k^c), 1 - \max_{i\leq v_1, k\leq v_2} \underline{P}(A_iB_k)\right\} \\
&= 1 - \min\left\{\min\left(1 - \underline{p}_{1w_1}, 1 - \underline{p}_{2w_2}\right), \max\left(0, \underline{p}_{1w_1} + \underline{p}_{2w_2} - 1\right)\right\} \\
&= \max\left(\underline{p}_{1w_1}, \underline{p}_{2w_2}\right).
\end{aligned}$$

The generalization on the case of n components is obvious. ■

Corollary 7 *If there is no information about independence of the system components and the probability distributions $F_i(t) = \Pr(X_i \leq t)$ of the component times to failure are known precisely, then*

$$\underline{R}^*(t) = \max_{i=1,\dots,n} F_i(t),\ \overline{R}^*(t) = \min\left(1, \sum_{i=1}^{n} F_i(t)\right).$$

Proof. The formulas follow directly from Theorem 3. ■

This means, even though the probability distributions of the component times to failure are known precisely and the judgement of the component independence is not introduced, then only the interval reliability measures can be found.

Corollary 8 *If the system components are identical, then there hold*

$$\underline{R}^*(t) = \underline{p}_w,\ \overline{R}^*(t) = \min\left(1, n\overline{p}_v\right).$$

Proof. The proof is obvious. ■

Corollary 9 *The lower system unreliability is defined only by probabilities of the largest intervals $[0, \alpha_{w_i}]$ which are included in $[0, t]$. The upper system unreliability is defined only by probabilities of the smallest intervals $[0, \alpha_{v_i}]$ which include $[0, t]$.*

Proof. The proof is obvious from Theorem 3. ■

3.2.2 Parallel system

Theorem 4 *If the system components are not judged to be independent, then the lower and upper bounds for the unreliability of the parallel system at time t are computed as follows:*

$$\underline{R}^*(t) = \max\left\{0, \sum_{i=1}^{n} \underline{p}_{iw_i} - (n-1)\right\},\ \overline{R}^*(t) = \min_{i=1,\dots,n} \overline{p}_{iv_i}.$$

Proof. Similarly to the proof of Theorem 3. Let us consider a parallel system consisting of two components for simplicity. Let us introduce notation: D is the event $\{\max(X_1, X_2) \leq t\}$, A_i is the event $\{X_1 \in [0, \alpha_{1i}]\}$ and A_i^c is the set complement to A_i, B_i is the event $\{X_2 \in [0, \alpha_{2i}]\}$. Then there hold

$$\begin{aligned}
\overline{R}^* &= \min\left\{\min_{i,k:D\subset A_iB_k} \overline{P}(A_iB_k), 1 - \max_{i,k:D^c\supset A_iB_k} \underline{P}(A_iB_k)\right\} \\
&= \min\left\{\overline{P}(A_{v_1}B_{v_2}), 1\right\} = \min\left\{\overline{p}_{v_1}, \overline{p}_{v_2}\right\}. \\
\underline{R}^* &= 1 - \overline{P}(D^c) = 1 - \min\{1, 1 - \underline{P}(A_{w_1}B_{w_2})\} \\
&= \underline{P}(A_{w_1}B_{w_2}) = \max\left(0, \underline{p}_{1w_1} + \underline{p}_{2w_2} - 1\right).
\end{aligned}$$

The generalization on the case of n components is obvious. ■

Corollary 10 *If there is no information about independence of the system components and the probability distributions $F_i(t) = \Pr(X_i \leq t)$ of the component times to failure are known precisely, then*

$$\underline{R}^*(t) = \max_{i=1,\dots,n}\left\{0, \sum_{i=1}^{n} F_i(t) - (n-1)\right\},\ \overline{R}^*(t) = \min_{i=1,\dots,n} F_i(t).$$

Proof. The formulas follow directly from Theorem 4. ■

Corollaries 7 and 10 state that the obtained expressions coincide with ones obtained for the case of completely known probability distributions and the lack of information about independence of components [8]. This means that Theorems 3 and 4 generalize the formulas for reliability to interval-valued probabilities.

Corollary 11 *If the system components are identical, then there hold*

$$\underline{R}^*(t) = \max\left\{0, n\underline{p}_w - (n-1)\right\}, \ \overline{R}^*(t) = \overline{p}_v.$$

Proof. The proof is obvious. ■

Corollary 12 *The lower system unreliability is defined only by probabilities of the largest intervals $[0, \alpha_{w_i}]$ which are included in $[0, t]$. The upper system unreliability is defined only by probabilities of the smallest intervals $[0, \alpha_{v_i}]$ which include $[0, t]$.*

Proof. The proof is obvious from Theorem 4. ■

4 Probabilities on nested intervals

Consider a case with the following partial information about probabilities of failures:

$$\underline{p}_{ij} \leq \Pr\{\underline{\alpha}_{ij} \leq X_i \leq \overline{\alpha}_{ij}\} \leq \overline{p}_{ij}, \ i = 1, ..., n, \ j = 1, ..., m_i, \tag{15}$$

where

$$[\underline{\alpha}_{i1}, \overline{\alpha}_{i1}] \subset [\underline{\alpha}_{i2}, \overline{\alpha}_{i2}] \subset ... \subset [\underline{\alpha}_{im_i}, \overline{\alpha}_{im_i}], \ i = 1, ..., n. \tag{16}$$

In other words, there are the nested intervals $[\underline{\alpha}_{ij}, \overline{\alpha}_{ij}]$ with the interval probabilities $[\underline{p}_{ij}, \overline{p}_{ij}]$ that the failure of the i-th component is inside these intervals, respectively. Here we have to note the additional condition $\overline{\alpha}_{i(m_i+1)} \to \infty$.

Let v_i be the maximal number j such that $\underline{\alpha}_{ij} \geq t$, i.e.,

$$v_i = \max\{j : \underline{\alpha}_{ij} \geq t\}.$$

Let w_i be the maximal number j such that $\overline{\alpha}_{ij} \leq t$, i.e.,

$$w_i = \max\{j : \overline{\alpha}_{ij} \leq t\}.$$

4.1 Independent components

4.1.1 Series systems

Theorem 5 *If the system components are statistically independent and governed by probabilities in the form of (15)-(16), then the lower and upper bounds for the unreliability of the series system at time t are computed as follows:*

$$\underline{R}(t) = 1 - \prod_{i=1}^{n}(1 - \underline{p}_{iw_i}), \ \overline{R}(t) = 1 - \prod_{i=1}^{n} \underline{p}_{iv_i}.$$

Proof. Let us consider a series system consisting of two components for simplicity. First, we assume that $\underline{p}_{ij} = \overline{p}_{ij} = p_{ij}$, for all i and j. Optimization problems (11)-(13) differ from optimization problems, corresponding to the case of nested intervals, with the following constraints:

$$\sum_{j=1}^{m_i+1} I_{[\underline{\alpha}_{ik},\overline{\alpha}_{ik}]}(x_j)c_j = p_{ik},\ i=1,2,\ k=1,...,m_i. \tag{17}$$

It can be shown similarly to the proof of Theorem 1 that the values x_{1k} and x_{2k} delivering the optima meet the following conditions: $x_{ik} \in [\underline{\alpha}_{ik},\overline{\alpha}_{ik}]\backslash[\underline{\alpha}_{i(k-1)},\overline{\alpha}_{i(k-1)}]$, $i=1,2$, for all possible k. It follows from these conditions and from (17) that

$$c_1 = p_{11},\ c_1 + c_2 = p_{12}, ..., \sum_{i=1}^{m_1} c_i = p_{1m_1},$$

$$d_1 = p_{21},\ d_1 + d_2 = p_{22}, ..., \sum_{i=1}^{m_2} d_i = p_{2m_2}.$$

Hence

$$c_k = p_{1k} - p_{1(k-1)},\ d_j = p_{2j} - p_{2(j-1)},\ k=1,...,m_1,\ j=1,...,m_2.$$

Then

$$\overline{R} = 1 - \underline{P}\left(\min(X_1,X_2) \geq t\right)$$

$$= 1 - \min \sum_{k=1}^{m_1+1} \sum_{j=1}^{m_2+1} I_{[t,\infty)}(\min(x_{1k},x_{2j}))c_k d_j.$$

Note that the objective function in the above expression achieves its minimum if there hold $I_{[t,\infty)}(\min(x_{2j},x_{1k})) = 0$ for all $k \leq m_1+1$ and $j \leq m_2+1$. However, there exist the values j and k such that $I_{[t,\infty)}(\min(x_{2j},x_{1k})) = 1$ for some combinations of x_{2j} and x_{1k}. Then $I_{[t,\infty)}(\min(x_{2j},x_{1k}))$ is always equal to 1 for all x_{2j} and x_{1k} such that $k \leq v_1$ and $j \leq v_2$. Thus, it can be concluded

$$\overline{R} = 1 - \sum_{k=1}^{v_1}\sum_{j=1}^{v_2} c_k d_j = 1 - p_{1v_1}p_{2v_2}.$$

Here $\overline{R}$ increases as p_{iv_i}, $i=1,...,n$, decrease. This implies that

$$\overline{R} = \max_{\underline{p}_{ij} \leq p_{ij} \leq \overline{p}_{ij}} \{1 - p_{1v_1}p_{2v_2}\} = 1 - \underline{p}_{1v_1}\underline{p}_{2v_2}.$$

The lower bound $\underline{R}$ can be similarly computed

$$\underline{R} = \underline{P}\left(\min(X_1,X_2) \leq t\right) = \min \sum_{k=1}^{m_1+1} \sum_{j=1}^{m_2+1} I_{[0,t]}(\min(x_{1k},x_{2j}))c_k d_j$$

$$= c_1(d_1 + ... + d_{m_2+1}) + ... + c_{w_1}(d_1 + ... + d_{m_2+1})$$

$$+ d_1(c_{w_1+1} + ... + c_{m_1+1}) + ... + d_{w_2}(c_{w_1+1} + ... + c_{m_1+1}) = 1 - (1 - p_{1w_1})(1 - p_{2w_2}).$$

Here $\underline{R}$ decreases as p_{iw_i}, $i=1,...,n$, decrease. This implies that

$$\underline{R} = \min_{\underline{p}_{ij} \leq p_{ij} \leq \overline{p}_{ij}} \{1 - (1 - p_{1w_1})(1 - p_{2w_2})\} = 1 - (1 - \underline{p}_{1w_1})(1 - \underline{p}_{2w_2}).$$

The generalization on the case of n components is obvious. ■

Corollary 13 *If the system components are identical, then there hold*

$$\underline{R}(t) = 1 - (1 - \underline{p}_w)^n, \ \overline{R}(t) = 1 - \underline{p}_v{}^n.$$

Proof. The proof is obvious. ■

4.1.2 Parallel systems

Theorem 6 *If the system components are statistically independent and governed by probabilities in the form of (15)-(16), then the lower and upper bounds for the unreliability of the parallel system at time t are computed as follows:*

$$\underline{R}(t) = \prod_{i=1}^{n} \underline{p}_{iw_i}, \ \overline{R}(t) = \prod_{i=1}^{n} (1 - \underline{p}_{iv_i}).$$

Proof. Similarly to the proof of Theorem 5. ■

It can be seen that the lower and upper bounds for the unreliability of series or parallel systems depend only on lower probabilities of nested intervals. This implies that knowledge of upper probabilities does not give any useful information.

Corollary 14 *If the system components are identical, then there hold*

$$\underline{R}(t) = \underline{p}_w{}^n, \ \overline{R}(t) = (1 - \underline{p}_v)^n.$$

Proof. The proof is obvious. ■

Corollary 15 *The lower (series and parallel) system unreliability is defined only by the probability of the largest interval $[\underline{\alpha}_w, \overline{\alpha}_w]$ which is included in $[0, t]$. The upper (series and parallel) system unreliability is defined only by the probability of the largest interval $[\underline{\alpha}_v, \overline{\alpha}_v]$ which does not include $[0, t]$.*

Proof. Obviously from Theorems 5, 6, and definitions of v and w. ■

4.2 Lack of knowledge about the independence of components

4.2.1 Series systems

Theorem 7 *If the information about the series system components is given in the form of (15)-(16), then there hold*

$$\underline{R}^*(t) = \max_{i=1,\dots,n} \underline{p}_{iw_i}, \ \overline{R}^*(t) = 1 - \max\left(0, \sum_{i=1}^{n} \underline{p}_{iv_i} - (n-1)\right). \tag{18}$$

Proof. Let us consider a series system consisting of two components for simplicity. Let us introduce notation: D is the event $\{\min(X_1, X_2) \geq t\}$, A_i is the event $\{X_1 \in [\underline{\alpha}_{1i}, \overline{\alpha}_{1i}] \backslash [\underline{\alpha}_{1(i-1)}, \overline{\alpha}_{1(i-1)}]\}$ and A_i^c is the set complement to A_i, B_i is the event $\{X_2 \in [\underline{\alpha}_{2i}, \overline{\alpha}_{2i}] \backslash [\underline{\alpha}_{2(i-1)}, \overline{\alpha}_{2(i-1)}]\}$, and $A_i B_k$ is a subset of the universal set $A_{m_1+1} \times B_{m_2+1}$. Then there hold

$$\overline{R}^* = 1 - \underline{P}(D) = 1 - \max\left\{ \max_{i,j: D \supset A_i B_j} \underline{P}(A_i B_j), 1 - \min_{i,j: D^c \subset A_i B_j} \overline{P}(A_i B_j) \right\}$$

$$= 1 - \max\left\{\max_{i\le v_1, j\le v_2} \max\left(\underline{p}_{1i} + \underline{p}_{2j} - 1, 0\right), 0\right\}$$

$$= 1 - \max\left\{0, \left(\underline{p}_{1v_1} + \underline{p}_{2v_2} - 1\right)\right\},$$

$$\underline{R}^* = 1 - \overline{P}(D) = 1 - \min\left\{\min_{i,k:D\subset A_iB_k} \overline{P}(A_iB_k), 1 - \max_{i,k:D^c\supset A_iB_k} \underline{P}(A_iB_k)\right\}$$

$$= 1 - \min\left\{1, 1 - \max_{i\le w_1, k\le m_2+1} \max_{k\le w_2, i\le m_1+1} \underline{P}(A_iB_k)\right\}$$

$$= \max\left\{\max(0, \underline{p}_{1w_1}), \max(0, \underline{p}_{2w_2})\right\} = \max\left(\underline{p}_{1w_1}, \underline{p}_{2w_2}\right).$$

The generalization on the case of n components is obvious. ■

Corollary 16 *If the information about the series system components is given as*

$$\underline{p}_i \le \Pr\{\underline{\alpha} \le X_i \le \overline{\alpha}\} \le \overline{p}_i,\ i = 1, ..., n,$$

then there hold for $t = \underline{\alpha}$

$$\underline{R}^*(t) = 0,\ \overline{R}^*(t) = 1 - \max\left(0, \sum_{i=1}^n \underline{p}_i - (n-1)\right),$$

and for $t = \overline{\alpha}$

$$\underline{R}^*(t) = \max_{i=1,\dots,n} \underline{p}_i,\ \overline{R}^*(t) = 1.$$

Proof. This is obvious from the following. If $t = \underline{\alpha} \le \overline{\alpha}$, then $v_i = \max\{j : \underline{\alpha}_{ij} \ge t\} = 1$ and $w_i = \max\{j : \overline{\alpha}_{ij} \le t\} = 0$ for all $i = 1, ..., n$. If $t = \overline{\alpha}$, then $v_i = 0$ and $w_i = 1$ for all $i = 1, ..., n$. ■

Corollary 17 *If the system components are identical, then there hold*

$$\underline{R}^*(t) = \underline{p}_w,\ \overline{R}^*(t) = 1 - \max\left(0, n\underline{p}_v - (n-1)\right).$$

Proof. The proof is obvious. ■

4.2.2 Parallel systems

Theorem 8 *If the information about the parallel system components is given in the form of (15)-(16), then there hold*

$$\underline{R}^*(t) = \max\left(0, \sum_{i=1}^n \underline{p}_{iw_i} - (n-1)\right),\ \overline{R}^*(t) = 1 - \max_{i=1,\dots,n} \underline{p}_{iv_i}. \tag{19}$$

Proof. Similarly to the proof of Theorem 7. ■

Corollary 18 *If the information about the parallel system components is given as*

$$\underline{p}_i \le \Pr\{\underline{\alpha} \le X_i \le \overline{\alpha}\} \le \overline{p}_i,\ i = 1, ..., n,$$

then there hold for $t = \underline{\alpha}$

$$\underline{R}^*(t) = 0,\ \overline{R}^*(t) = 1 - \max_{i=1,\dots,n} \underline{p}_i,$$

and for $t = \overline{\alpha}$

$$\underline{R}^*(t) = \max\left(0, \sum_{i=1}^n \underline{p}_i - (n-1)\right),\ \overline{R}^*(t) = 1.$$

Proof. Similarly to the proof of Corollary 16. ■

Corollary 19 *If the system components are identical, then there hold*

$$\underline{R}^{*}(t)=\max\left(0,n\underline{p}_{w}-(n-1)\right),\ \overline{R}^{*}(t)=1-\underline{p}_{v}.$$

Proof. The proof is obvious. ■

Corollary 20 *The lower (series and parallel) system unreliability is defined only by the probability of the largest interval* $[\underline{\alpha}_w,\overline{\alpha}_w]$ *which is included in* $[0,t]$. *The upper (series and parallel) system unreliability is defined only by the probability of the largest interval* $[\underline{\alpha}_v,\overline{\alpha}_v]$ *which does not include* $[0,t]$.

Proof. Obviously from Theorems 7, 8, definitions of v and w. ■

Theorem 9 *If initial information about the series or parallel systems is represented as a set of probabilities defined on nested intervals (for the cases of independence and the lack of information about independence of components), then either* $\underline{R}(t)=0$ *or* $\overline{R}(t)=1$.

Proof. Consider the series system and the case of independent components. It follows from the definition of v_i and w_i that if $v_i>0$, then $w_i=0$ and $\underline{p}_{iw_i}=0$, if $w_i>0$, then $v_i=0$ and $\underline{p}_{iv_i}=0$. Suppose that there hold $\underline{p}_{iv_i}\geq 0$ (all $w_i=0$ and $\underline{p}_{iw_i}=0$) for all $i=1,...,n$. Then it follows from Theorem 5 that

$$\underline{R}(t)=0,\ \overline{R}(t)=1-\prod_{i=1}^{n}\underline{p}_{iv_i}\leq 1.$$

Now suppose that there holds $\underline{p}_{iw_i}\geq 0$ for at least one value of i. In this case, there exists at least one probability $\underline{p}_{iv_i}=0$ and there hold

$$\underline{R}(t)=1-\prod_{i=1}^{n}(1-\underline{p}_{iw_i})\geq 0,\ \overline{R}(t)=1,$$

as was to be proved for the case of independent components. The same can be similarly proved for the parallel system.

Now we consider the case of the lack of information about independence. The condition of independence is some additional information about the system. This implies that $\underline{R}(t)\geq\underline{R}^{*}(t)$ and $\overline{R}(t)\leq\overline{R}^{*}(t)$. Then it is obvious that if $\underline{R}(t)=0$, then $\underline{R}^{*}(t)=0$, and if $\overline{R}(t)=1$, then $\overline{R}^{*}(t)=1$. ■

We have considered the most important special cases of initial information when points of probability distributions of the component times to failure or probabilities defined on nested intervals are known. The reliability assessments for these cases have been obtained in the explicit form. If initial information is represented in the form of lower and upper probabilities defined on arbitrary intervals, then solutions to optimization problems (1)-(2) and (3)-(4) can be obtained only in the numerical form by means of special numerical methods, for example, approximation of the aforementioned optimization problems by linear programming ones.

5 Formal duality of results

Let us bring together all main results in Table 1. Here the notation I and L mean that expressions for the unreliability in the corresponding rows in Table 1 are given for the case of independent components (I) and for the case of the lack of information about independence (L).

The definition of duality for systems can be written as follows [2].

Table 1: The list of results for different cases of initial data

	Series system		Parallel system	
	lower bound	upper bound	lower bound	upper bound
	Partially known probability distributions			
	$v_i = \min\{j : \alpha_{ij} \geq t\},\ w_i = \max\{j : \alpha_{ij} \leq t\}$			
I	$1 - \prod_{i=1}^{n}(1 - \underline{p}_{iw_i})$	$1 - \prod_{i=1}^{n}(1 - \overline{p}_{iv_i})$	$\prod_{i=1}^{n} \underline{p}_{iw_i}$	$\prod_{i=1}^{n} \overline{p}_{iv_i}$
L	$\max_{i=1,\dots,n} \underline{p}_{iw_i}$	$\min\left(1, \sum_{i=1}^{n} \overline{p}_{iv_i}\right)$	$1 - \min\left\{1, n - \sum_{i=1}^{n} \underline{p}_{iw_i}\right\}$	$\min_{i=1,\dots,n} \overline{p}_{iv_i}$
	Probabilities on nested intervals			
	$v_i = \max\{j : \underline{\alpha}_{ij} \geq t\},\ w_i = \max\{j : \overline{\alpha}_{ij} \leq t\}$			
I	$1 - \prod_{i=1}^{n}(1 - \underline{p}_{iw_i})$	$1 - \prod_{i=1}^{n} \underline{p}_{iv_i}$	$\prod_{i=1}^{n} \underline{p}_{iw_i}$	$\prod_{i=1}^{n}(1 - \underline{p}_{iv_i})$
L	$\max_{i=1,\dots,n} \underline{p}_{iw_i}$	$\min\left(1, n - \sum_{i=1}^{n} \underline{p}_{iv_i}\right)$	$1 - \min\left(1, n - \sum_{i=1}^{n} \underline{p}_{iw_i}\right)$	$1 - \max_{i=1,\dots,n} \underline{p}_{iv_i}$

Definition 3 *Given a system with the time to failure $g(\mathbf{X})$, then its dual system has the time to failure $g^D(\mathbf{X}) = 1 - g(1 - \mathbf{X})$, where $1 - \mathbf{X} = (1 - x_1, ..., 1 - x_n)$.*

According to this definition, series and parallel systems are dual. In the case of precise and complete initial data, one can conclude about the system unreliability

$$R(p_1, ..., p_n) = 1 - R^D(1 - p_1, ..., 1 - p_n),$$

where R and R^D are the unreliabilities of a system and its dual system as functions of the component unreliabilities $p_1, ..., p_n$, respectively.

By analyzing Table 1, we can give another conclusion about reliability of dual systems taking into account incompleteness of available information. If two systems are dual, then

$$\underline{R}(\underline{\mathbf{P}}, \overline{\mathbf{P}}, \mathbf{W}) = 1 - \overline{R}^D(1 - \overline{\mathbf{P}}, 1 - \underline{\mathbf{P}}, \mathbf{V}),$$

$$\overline{R}(\underline{\mathbf{P}}, \overline{\mathbf{P}}, \mathbf{V}) = 1 - \underline{R}^D(1 - \overline{\mathbf{P}}, 1 - \underline{\mathbf{P}}, \mathbf{W}),$$

where $\underline{\mathbf{P}} = (\underline{p}_1, ..., \underline{p}_n)$, $\overline{\mathbf{P}} = (\overline{p}_1, ..., \overline{p}_n)$, $\mathbf{W} = (w_1, ..., w_n)$, $\mathbf{V} = (v_1, ..., v_n)$.

So, by knowing the reliability of a system under incomplete information, we can always write expressions for reliability of its dual system. For example, suppose that the lower bound for the unreliability of a series system by the lack of information about independence is determined as $\underline{R} = \max_{i=1,...,n} \underline{p}_{iw_i}$. Then we can write the upper bound for the unreliability of a parallel system under the same initial information as

$$\overline{R} = 1 - \max_{i=1,...,n}(1 - \overline{p}_{iv_i}) = \min_{i=1,...,n} \overline{p}_{iv_i}.$$

6 Numerical example

Let us consider a series system consisting of two components. Suppose that times to failure of the first and the second components are governed by exponential distributions with mean times to failure 100 hours and 50 hours, respectively. However, we assume that the true lifetime distributions are unknown and there is only the following information about the first component:

$$\alpha_{11} = 50,\ \underline{p}_{11} = \overline{p}_{11} = 0.39,$$

and about the second component:

$$\alpha_{21} = 30,\ \underline{p}_{21} = \overline{p}_{21} = 0.45,\ \alpha_{22} = 70,\ \underline{p}_{21} = \overline{p}_{21} = 0.75.$$

In other words, we know only one point of the probability distribution of the first component lifetime and two points of the probability distribution of the second component lifetime. It should be noted that the probabilities are taken in accordance with the given exponential distributions. This is important for comparison of computational results below.

Let us find the probability of the system failure before time 40 hours, i.e., $R(40)$. In this case, there hold $v_1 = 1$, $v_2 = 2$, $w_1 = 0$, $w_1 = 1$.

1. Independent components (Theorem 1):

$$\underline{R}(40) = 1 - (1 - \underline{p}_{10})(1 - \underline{p}_{21}) = 1 - (1 - 0)(1 - 0.45) = 0.45,$$
$$\overline{R}(40) = 1 - (1 - \overline{p}_{11})(1 - \overline{p}_{22}) = 1 - (1 - 0.39)(1 - 0.75) = 0.847.$$

2. The lack of information about independence (Theorem 3):

$$\underline{R}^*(40) = \max\left(\underline{p}_{10}, \underline{p}_{21}\right) = \max(0, 0.45) = 0.45,$$
$$\overline{R}^*(40) = \min(1, \overline{p}_{11} + \overline{p}_{22}) = \min(1, 0.39 + 1) = 1.$$

Let us carry out the calculation under condition of known distributions of the component lifetimes.

1. Independent components (Corollary 1):

$$\underline{R}(40) = \overline{R}(40) = 1 - \exp(-0.01 \cdot 40) \cdot \exp(-0.02 \cdot 40) = 0.699.$$

2. The lack of information about independence (Corollary 7):

$$\underline{R}^*(40) = \max(1 - \exp(-0.01 \cdot 40), 1 - \exp(-0.02 \cdot 40)) = 0.55,$$
$$\overline{R}^*(40) = \min(1, 1 - \exp(-0.01 \cdot 40) + 1 - \exp(-0.02 \cdot 40)) = 0.88.$$

The above example illustrates the impact of incompleteness of initial data on the resulting reliability measures.

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