Reliability models of m-out-of-n systems under incomplete information

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Abstract

Most methods of reliability analysis of m-out-of-n systems assume that the precise probability distributions of the component time to failure are available and the system components are independent. However, this assumption may be unreasonable in a wide scope of cases (software, human-machine systems). Therefore, imprecise reliability models of m-out-of-n systems are proposed in the paper under condition that there is only some partial information about component lifetime distributions including points of unknown distributions, probabilities on nested intervals and interval mean times to failure. It is shown how the reliability assessments may vary with the type of the available information. The impact of the component independence condition on the reliability of systems is studied.

Keywords: reliability, m-out-of-n system, imprecise probability theory, possibility measure, probability distribution, independence, mean time to failure, optimization problem

1. Introduction

In order to increase the reliability of a system and to design highly reliable systems, the redundancy technique is usually used. One of the most common forms of redundancy is the m-out-of-n system. The m-out-of-n systems have received very much attention in the past years. Applications of m-out-of-n systems can be found in various applied areas, for example, in safety monitoring, N version programming, etc. Therefore, a number of articles were devoted to algorithms and methods for reliability analysis of such the systems [2–5].

However, most results related to m-out-of-n systems assume that there is complete information about a system and its components, that is, (i) all probabilities are precise and perfectly determinable, (ii) the system components are independent. If the information we have about the functioning of components and a system is based on a statistical analysis, then a probabilistic uncertainty model should be used in order to mathematically represent and manipulate that information. However, the reliability assessments that are combined to describe the system and components may come from various sources. Some assessments may be objective measures based on relative frequencies or on well established statistical models. A part of the reliability assessments may be supplied by experts. Assessments may be provided by a user of the system during the experimental service. This implies that very often only some partial information is available and the first condition of the information completeness is violated.

It should be noted that it is difficulty to expect the independence of components in many applications.

*This work was supported by the Alexander von Humboldt Foundation (Germany).

Let us consider the N version programming. If these programs were developed by means of the same programming language, then possible errors in a language library of typical functions produce dependent faults in both programs. Several experimental studies show [6] that the assumption of independence of failures between independently developed programs does not hold. Moreover, the main difficulty here is that the degree of dependency is unknown. The same examples can be presented for various applications. This implies that the second condition of the information completeness is also violated.

In order to solve the problem of use of partial information and to take into account the lack of information about independency of components, the theory of imprecise probabilities (also called the theory of lower previsions [1], the theory of interval statistical models [7], the theory of interval probabilities [8,9]) can be successfully applied. A general framework for the theory of imprecise probabilities is provided by upper and lower previsions. They can model a very wide variety of kinds of uncertainty, partial information, and ignorance. The rules used in the theory of lower previsions, which are based on a general procedure called natural extension, can be applied to various measures.

The imprecise reliability models have been considered in the literature [10–13]. In this paper, we consider reliability analysis of m-out-of-n systems under partial information about probabilities of times to failure of the system components and about the component mean times to failure (MTTFs). The most expressions are obtained in the explicit form.

2. Preliminary definitions

Consider a system consisting of n components. Let X_i be the time to failure of the i-th component, $i = 1,...,n$. Denote $\mathcal{X} = \{X_1,...,X_n\}$ and $\mathbf{X} = (x_1,...,x_n)$. Let $X_{\min,1} = \min\mathcal{X}$, $X_{\min,2} = \min\mathcal{X}\backslash X_{\min,1}$,..., $X_{\min,m+1} = \min\mathcal{X}\backslash\{X_{\min,1}, X_{\min,2},...,X_{\min,m}\}$. Then the time to failure of the m-out-of-n system is determined as $g(\mathcal{X}) = X_{\min,m+1}$. From the statistical point of view, $X_{\min,m+1}$ is an order statistics from the sample $\mathcal{X}$. It should be noted that series ($g(\mathcal{X}) = \min_{i=1,...,n} X_i$) and parallel ($g(\mathcal{X}) = \max_{i=1,...,n} X_i$) systems can be regarded as special cases of the m-out-of-n system when $m = 0$ and $m = n-1$, respectively.

Let $\varphi_{ij}(x_i)$ be a function of the random time to failure X_i of the i-th component. According to [14], the system lifetime can be uniquely determined by the component lifetimes. Then there exists a function $g(\mathcal{X})$ of the component lifetimes characterizing the system reliability behavior. Suppose that partial information is represented as a set of lower and upper previsions (expectations) $\underline{a}_{ij} = \underline{\mathbb{E}}\varphi_{ij}(X_i)$ and $\overline{a}_{ij} = \overline{\mathbb{E}}\varphi_{ij}(X_i)$, $i = 1,...,n$, $j = 1,...,m_i$, of functions φ_{ij}. Here m_i is the number of judgements that are related to the i-th component. For example, an interval-valued probability of failure in the interval $[a, b]$ can be represented as previsions of the indicator function $I_{[a,b]}(X)$ such that $I_{[a,b]}(X) = 1$ if $X \in [a, b]$ and $I_{[a,b]}(X) = 0$ if $X \notin [a, b]$. A system reliability measure can be regarded as the expectation $\mathbb{E}g$ of the function g

$$\mathbb{E}g = \int_{\mathbb{R}^n_+} g(\mathbf{X})\rho(\mathbf{X})\mathrm{d}\mathbf{X},$$

where ρ is a probability density function of the system time to failure.

However, we do not know the density ρ because our initial information is restricted only by the lower and upper expectations $\underline{\mathbb{E}}\varphi_{ij}$ and $\overline{\mathbb{E}}\varphi_{ij}$, $i = 1,...,n$, $j = 1,...,m_i$, and there is no information about distributions of the component time to failure. At the same time, the available lower and upper expectations form a set $\mathcal{P}$ of possible densities that are consistent with these expectations. This means that we can find only the largest and smallest possible values of $\mathbb{E}g$ for all densities from the set $\mathcal{P}$. It can be carried out by solving the following optimization problems (also called natural extension):

$$\underline{\mathbb{E}}g = \min_{\mathcal{P}} \int_{\mathbb{R}^n_+} g(\mathbf{X})\rho(\mathbf{X})\mathrm{d}\mathbf{X},\ \overline{\mathbb{E}}g = \max_{\mathcal{P}} \int_{\mathbb{R}^n_+} g(\mathbf{X})\rho(\mathbf{X})\mathrm{d}\mathbf{X}, \quad (1)$$

subject to

$$\underline{a}_{ij} \leq \int_{\mathbb{R}_+^n} \varphi_{ij}(x_i)\rho(\mathbf{X})\mathrm{d}\mathbf{X} \leq \overline{a}_{ij},\ i \leq n,\ j \leq m_i. \tag{2}$$

Here the minimum and maximum are taken over the set $\mathcal{P}$ of all possible n-dimensional joint density functions $\{\rho(\mathbf{X})\}$ satisfying conditions (2). The obvious constraints for densities ρ to the optimization problems such that $\rho(\mathbf{X}) \geq 0$, $\int_{\mathbb{R}_+^n} \rho(\mathbf{X})\mathrm{d}\mathbf{X} = 1$ will not be written.

It should be noted that only joint densities are used in optimization problems (1)-(2) because in a general case we may not be aware whether the variables $X_1, ..., X_n$ are dependent or not. If it is known that components are independent, then $\rho(\mathbf{X}) = \rho(x_1) \cdots \rho(x_n)$. Here we use the definition of independence in the sense of classical probability theory. In this case, the set $\mathcal{P}$ is reduced and consists only of the densities that can be represented as a product of marginal ones. The optimization problem for computing, for instance, a new lower prevision is of the form:

$$\underline{\mathbb{E}}g = \min_{\mathcal{P}} \int_{\mathbb{R}_+^n} g(\mathbf{X})\rho_1(x_1) \cdots \rho_n(x_n)\mathrm{d}\mathbf{X}, \tag{3}$$

subject to

$$\underline{a}_{ij} \leq \int_{\mathbb{R}_+} \varphi_{ij}(x_i)\rho_i(x_i)\mathrm{d}x_i \leq \overline{a}_{ij},\ i \leq n,\ j \leq m_i. \tag{4}$$

In this case, the problem is non-linear and difficult to be solved.

The reliability analysis of m-out-of-n systems for some important special cases of the initial information is considered in the paper. At that, the calculated reliability measure is the probability $R(t)$ that the system time to failure less than time t, i.e. $R(t) = \Pr\{g(\mathcal{X}) \leq t\}$. This measure is called the unreliability and can be represented also as the expectation of the indicator function $I_{[0,t]}(g)$. The reliability $Q(t) = \Pr\{g(\mathcal{X}) \geq t\}$ can be found from the condition $Q(t) = 1 - R(t)$. It can be represented as the expectation of the indicator function $I_{[t,\infty)}(g)$. If the system reliability measures are interval-valued, then $\overline{Q}(t) = 1 - \underline{R}(t)$, and $\underline{Q}(t) = 1 - \overline{R}(t)$. Another reliability measure is the MTTF T of the system and its bounds. It can be represented as the expectation of the function g.

3. Partially known probability distributions

Assume that initial information about the time to failure X of the system components is given in the following form:

$$\underline{p}_j \leq \Pr\{X \leq \alpha_j\} \leq \overline{p}_j,\ j = 1, ..., l. \tag{5}$$

Here it is assumed that $\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_l$. This assumption is obvious because $\underline{p}_j$, $\overline{p}_j$, $j = 1, ..., l$, are values of interval probability distributions. In other words, only l points of a probability distribution of X are known with some accuracy. For processing the above information, it has to be consistent. If there is the partial information in the form of (5), then the condition of consistency of probabilities is

$$\underline{p}_i \leq \overline{p}_j,\ \forall i \leq j. \tag{6}$$

3.1. Independent components

Suppose variables X_i, $i = 1, ..., n$ are independent. Then problems (1)-(2) can be rewritten as

$$\underline{R}(t)(\overline{R}(t)) = \min_{\mathcal{P}}(\max_{\mathcal{P}}) \int_{\mathbb{R}_+^n} I_{[0,t]}(g(\mathbf{X}))\rho(x_1) \cdots \rho(x_n)\mathrm{d}\mathbf{X}, \tag{7}$$

subject to

$$\underline{p}_j \leq \int_{\mathbb{R}_+} I_{[0,\alpha_j]}(x)\rho(x)\mathrm{d}x \leq \overline{p}_j,\ j=1,...,l. \tag{8}$$

Without loss of generality, it is assumed $\underline{p}_0 = \overline{p}_0 = 0$, $\underline{p}_{l+1} = \overline{p}_{l+1} = 1$, $\alpha_0 = 0$, $\alpha_{l+1} \to \infty$.

Let v be the minimal number j such that $\alpha_j \geq t$, i.e. $v = \min\{j : \alpha_j \geq t\}$, and let w be the maximal number j such that $\alpha_j \leq t$, i.e. $w = \max\{j : \alpha_j \leq t\}$.

Theorem 1 *If the system components are statistically independent, identical and governed by the partially known consistent probability distribution in the form of (5), then the lower and upper bounds for the unreliability at time t of the m-out-of-n system are computed as follows:*

$$\underline{R}(t) = \sum_{i=m+1}^{n} \binom{n}{i} \underline{p}_w^i (1-\underline{p}_w)^{n-i},\ \overline{R}(t) = \sum_{i=m+1}^{n} \binom{n}{i} \overline{p}_v^i (1-\overline{p}_v)^{n-i}. \tag{9}$$

Proof. Let us consider the 1-out-of-3 system for simplicity. First, we assume that $\underline{p}_i = \overline{p}_i = p_i$ for all $i=1,...,l$. It was proven in [15] that solutions for optimization problems similar to (7)-(8) exist on special types of distributions (see Appendix), and referring to this property the following optimization problem, equivalent to the above, can be stated:

$$\underline{R}(\overline{R}) = \min\,(\max) \sum_{i=1}^{l+1}\sum_{k=1}^{l+1}\sum_{j=1}^{l+1} I_{[0,t]}(g(x_i,x_k,x_j))c_i c_k c_j, \tag{10}$$

subject to

$$\sum_{k=1}^{l+1} c_k = 1,\ \sum_{i=1}^{l+1} I_{[0,\alpha_k]}(x_i)c_i = p_k,\ k=1,...,l. \tag{11}$$

Here the minimum and maximum are taken over the set of variables $x_i, c_i \in \mathbb{R}_+$, $i=1,...,l$, subject to the constraints. Assume that $x_1 \leq x_2 \leq ... \leq x_{l+1}$ are the values delivering min and max to the unreliability computed according to objective function (10). Let us prove that values x_k delivering the optima meet the following conditions for all possible k: $x_k \in [\alpha_{k-1}, \alpha_k]$. Suppose that there are two optimal values of x_j and x_k such that $x_j \in [\alpha_{k-1}, \alpha_k]$ and $x_k \in [\alpha_{k-1}, \alpha_k]$. This implies that if $j<k$, then it follows from (11) that the interval $[\alpha_{j-1}, \alpha_j]$ is empty and numbers of points x_i in intervals $[0, \alpha_{k-2}]$ and $[0, \alpha_{k-1}]$ are the same. In other words, for computing the probabilities p_{k-2} and p_{k-1}, we have the same non-zero values of indicator functions in (11) and there hold

$$c_1 + ... + c_{j-1} = p_{k-2},\ c_1 + ... + c_{j-1} = p_{k-1},$$

which is a contradiction. Similarly, if $j>k$, then it follows from (11) that

$$c_1 + ... + c_{k-1} = p_{j-1},\ c_1 + ... + c_{k-1} = p_j,$$

which is also a contradiction.

Similarly, we arrive at contradictions for an arbitrary number of values x_k belonging to the same interval. This implies that $x_k \in [\alpha_{k-1}, \alpha_k]$. It follows from these conditions ($x_k \in [\alpha_{k-1}, \alpha_k]$) and from (11) that

$$c_1 = p_1,\ c_1 + c_2 = p_2, ..., \sum_{i=1}^{l} c_i = p_l.$$

Hence $c_k = p_k - p_{k-1}$, $k = 1, ..., l$.

Note that function (10) achieves its minimum if there holds $I_{[0,t]}(g(x_i, x_k, x_j)) = 0$ for all $k \leq l+1$, $j \leq l+1$, $i \leq l+1$. However, there exist values i, j, k such that $I_{[0,t]}(g(x_i, x_k, x_j)) = 1$ for some combinations of x_i, x_k, x_j. It is necessary to find all cases when $I_{[0,t]}(g(x_i, x_k, x_j)) = 1$ for all values x_i, x_k, x_j from intervals $[\alpha_{i-1}, \alpha_i]$, $[\alpha_{k-1}, \alpha_k]$, $[\alpha_{j-1}, \alpha_j]$.

Let us denote for clarity $b_i = c_i$, $d_j = c_j$. Since the system fails if $m+1$ components fail, then at least $m+1$ of all different x_i must be less than t. This implies that

$$\begin{aligned}
\underline{R} &= b_1 c_1 (d_{w+1} + ... + d_{l+1}) + ... + b_1 c_w (d_{w+1} + ... + d_{l+1}) + ... \\
&+ b_w c_1 (d_{w+1} + ... + d_{l+1}) + ... + b_w c_w (d_{w+1} + ... + d_{l+1}) + ... \\
&+ b_1 c_1 (d_1 + ... + d_w) + ... + b_1 c_w (d_1 + ... + d_w) \\
&= \sum_{i=1}^{w} b_i \sum_{k=1}^{w} c_i \sum_{j=w+1}^{l+1} d_i + \sum_{i=1}^{w} b_i \sum_{k=w+1}^{l+1} c_i \sum_{j=1}^{w} d_i \\
&+ \sum_{i=w+1}^{l+1} b_i \sum_{k=1}^{w} c_i \sum_{j=1}^{w} d_i + \sum_{i=1}^{w} b_i \sum_{k=1}^{w} c_i \sum_{j=1}^{w} d_i.
\end{aligned}$$

Note that

$$\sum_{i=w+1}^{l+1} b_i = \sum_{j=w+1}^{l+1} d_i = \sum_{k=w+1}^{l+1} c_i = \sum_{j=w+1}^{l+1} (p_j - p_{j-1}) = 1 - p_w,$$

$$\sum_{i=1}^{w} b_i = \sum_{j=1}^{w} d_i = \sum_{k=1}^{w} c_i = \sum_{k=1}^{w} (p_j - p_{j-1}) = p_w.$$

Then

$$\underline{R} = 3p_w^2 (1 - p_w) + p_w^3 = \sum_{i=2}^{3} \binom{3}{i} p_w^i (1 - p_w)^{3-i}.$$

The generalization on the case of n components is obvious and this completes the proof of the lower bound.

Consider the upper probability. The objective function in this case can be represented as follows:

$$\overline{R} = 1 - \min \sum_{i=1}^{l+1} \sum_{k=1}^{l+1} \sum_{j=1}^{l+1} I_{[t,\infty)}(g(x_i, x_k, x_j)) c_i c_k c_j.$$

Since the system is in a working state if at least $n - m$ components are working, then at least $n - m$ of all different x_i must be larger than t. Similarly to the proof for the lower bound, we obtain

$$\begin{aligned}
\overline{R} &= 1 - \sum_{i=n-m}^{n} \binom{n}{i} \left(\sum_{k=v+1}^{l+1} c_k \right)^i \left(\sum_{k=1}^{v} c_k \right)^{n-i} \\
&= \sum_{i=0}^{n-m-1} \binom{n}{n-i} (1 - p_v)^i p_v^{n-i} = \sum_{i=m+1}^{n} \binom{n}{i} p_v^i (1 - p_v)^{n-i}.
\end{aligned}$$

Note that the functions $\underline{R}(t)$ and $\overline{R}(t)$ increase with p_w and p_v due to condition of monotony of the m-out-of-n system. This implies that the lower bound $\underline{R}(t)$ is determined by the lower probability $\underline{p}_w$, and the upper bound $\overline{R}(t)$ is determined by the upper probability $\overline{p}_v$, as was to be proved. ■

Corollary 1 *If the system components are statistically independent and the probability distribution* $F(t) = \Pr(X \leq t)$ *of the time to component failure is known precisely, then*

$$\underline{R}(t) = \overline{R}(t) = \sum_{i=m+1}^{n} \binom{n}{i} F^i(t)(1-F(t))^{n-i}.$$

Proof. It follows from Theorem 1 and from the conditions $v = \min\{j : \alpha_j \geq t\}$, $w = \max\{j : \alpha_j \leq t\}$, $\alpha_j = t$, that $v = w$. Then $p_v = p_w = F(t)$, and the proof is obvious. ■

Corollary 2 *The lower system unreliability is defined only by the probability of the largest interval* $[0, \alpha_w]$ *which is included in* $[0, t]$*. The upper system unreliability is defined only by the probability of the smallest interval* $[0, \alpha_v]$ *which includes* $[0, t]$*.*

Proof. The proof is obvious from (9) and definitions of v and w. ■

Corollary 3 *There hold for the series system*

$$\underline{R}(t) = 1 - (1 - \underline{p}_w)^n,\ \overline{R}(t) = 1 - (1 - \overline{p}_v)^n,$$

and for the parallel system

$$\underline{R}(t) = \underline{p}_w{}^n,\ \overline{R}(t) = \overline{p}_v{}^n.$$

Proof. The expressions for the series and parallel systems can be obtained from Theorem 1 and conditions $m = 0$ and $m = n - 1$, respectively. ■

3.2. Lack of knowledge about independence of components

It was assumed in the previous section that the system components are independent. Now we remove this additional assumption and suppose that there is no information about independence of components. The asterisk notation in $\underline{R}^*$ and $\overline{R}^*$ will mean that the bounds for the unreliability are obtained based on the lack of information about independence of components.

Theorem 2 *If the system components are not judged to be independent, then the lower and upper bounds for the unreliability at time* t *of the* m*-out-of-*n *system are computed as follows:*

$$\underline{R}^*(t) = \max\left\{0, (m+1)\underline{p}_w - m\right\},\ \overline{R}^*(t) = \min\left(1, (n-m)\overline{p}_v\right).$$

Proof. Let us consider the 1-out-of-3 system for simplicity. Let us introduce notation: D is the event $\{g(X_1, X_2, X_2) \leq t\}$, A_i is the event $\{X_i \in [0, \alpha_i]\}$ and A_i^c is the set complement to A_i, $A_i A_k$ is a subset of the universal set $A_{l+1} \times A_{l+1}$. According to [7], for computing lower and upper probabilities of an event D on the basis of available probabilities of some events A, the following expressions, which can be regarded as a special case of (1)-(2) when the functions φ_{ij} and g are indicator functions of events, can be written:

$$\underline{P}(D) = \max\left\{\max_{D \supset A} \underline{P}(A), 1 - \min_{D^c \subset A} \overline{P}(A)\right\},$$

$$\overline{P}(D) = \min\left\{\min_{D \subset A} \overline{P}(A), 1 - \max_{D^c \supset A} \underline{P}(A)\right\}.$$

Here D and A are arbitrary events. The same expressions can be found in [16]. Since the above expressions are special cases of optimization problems (1)-(2), then the resulting bounds for the unreliability are tight.

By using the reasoning similarly to the proof of Theorem 1 and by using the introduced notation, the above expressions can be rewritten as

$$\underline{R}^* = \underline{P}(D) = \max\left\{ \max_{i,j,k: D\supset A_iA_jA_k} \underline{P}(A_iA_jA_k), 1 - \min_{i,j,k: D^c\subset A_iA_jA_k} \overline{P}(A_iA_jA_k)\right\}.$$

Here the condition $D^c \subset A_iA_jA_k$ is valid only if $A_i = A_j = A_k = A_{l+1}$. This implies that

$$\min_{i,j,k: D^c\subset A_iA_jA_k} \overline{P}(A_iA_jA_k) = \overline{P}(A_{l+1}A_{l+1}A_{l+1}) = 1.$$

By using the fact that the system fails if at least $m+1$ components fail, we can conclude that the condition $D \supset A_iA_jA_k$ is valid for the following combinations of sets:

$$\begin{aligned} &i \le w, j \le w, k \le w; \quad i \le w, j \le w, k \le l+1; \\ &i \le w, j \le l+1, k \le w; \quad i \le l+1, j \le w, k \le w. \end{aligned}$$

It should be noted that the 2-nd, 3-rd and 4-th combinations have the same probability because probabilities of failures are identical. This implies that

$$\begin{aligned} \max_{i,j,k: D\supset A_iA_jA_k} \underline{P}(A_iA_jA_k) &= \max\left\{ \max_{i\le w, j\le w, k\le w} \underline{P}(A_iA_jA_k), \max_{i\le w, j\le w, k\le l+1} \underline{P}(A_iA_jA_k)\right\} \\ &= \max\left\{ \max\left(0,\ 3\underline{p}_w - 2\right), \max\left(0,\ 2\underline{p}_w + 1 - 2\right)\right\}. \end{aligned}$$

Since

$$(2\underline{p}_w + 1) - (3\underline{p}_w) = 1 - \underline{p}_w \ge 0,$$

then $\max_{i,j,k: D\supset A_iA_jA_k} \underline{P}(A_iA_jA_k) = \max\left(0,\ 2\underline{p}_w + 1 - 2\right)$.

By generalizing the results on arbitrary m and n, we obtain

$$\max_{i,\dots,k: D\supset A_i\cdots A_k} \underline{P}(A_i \cdots A_k) = \max\left(0,\ (m+1)\underline{p}_w - m\right).$$

Hence

$$\underline{R}^* = \max\left\{ \max\left(0,\ (m+1)\underline{p}_w - m\right), 1 - 1\right\} = \max\left(0,\ (m+1)\underline{p}_w - m\right).$$

The upper bound can be obtained in the same way:

$$\overline{R}^* = \overline{P}(D) = \min\left\{ \min_{i,j,k: D\subset A_iA_jA_k} \overline{P}(A_iA_jA_k), 1 - \max_{i,j,k: D^c\supset A_iA_jA_k} \underline{P}(A_iA_jA_k)\right\}.$$

The condition $D \subset A_iA_jA_k$ is valid if $A_i = A_j = A_k = A_{l+1}$. Then $\min_{i,j,k: D\subset A_iA_jA_k} \overline{P}(A_iA_jA_k) = 1$. By using the fact that the system is working if at least $n-m$ components are working, we can conclude that

$$\begin{aligned} \max_{i,j,k: D^c\supset A_iA_jA_k} \underline{P}(A_iA_jA_k) &= \max\left\{ \max_{i\ge v, j\ge v, k\ge v} \underline{P}(A_i^cA_j^cA_k^c), \max_{i\ge v, j\ge v, k\ge 0} \underline{P}(A_i^cA_j^cA_k^c)\right\} \\ &= \max\left\{ \max\left(0,\ 3(1-\overline{p}_v) - 2\right), \max\left(0,\ 2(1-\overline{p}_v) - 1\right)\right\}. \end{aligned}$$

Since

$$(\,2(1-\overline{p}_v) - 1) - (3(1-\overline{p}_v) - 2) = \overline{p}_v \ge 0,$$

then $\max_{i,j,k:D^c\supset A_iA_jA_k} \underline{P}(A_iA_jA_k) = \max(0,\ 2(1-\overline{p}_v)-1)$.

By generalizing the results on arbitrary m and n, we obtain

$$\max_{i,j,k:D^c\supset A_i\cdots A_k} \underline{P}(A_i\cdots A_k) = \max\left(0,\ (n-m)(1-\overline{p}_v)-(n-m-1)\right).$$

Hence

$$\begin{aligned}\overline{R}^* &= \min\left\{1, 1-\max\left(0,\ (n-m)(1-\overline{p}_v)-(n-m-1)\right)\right\} \\ &= \min\left\{1,\ (n-m)\overline{p}_v\right\},\end{aligned}$$

as was to be proved. ■

Corollary 4 *If there is no information about independence of the system components and the probability distribution $F(t)=\Pr(X\leq t)$ of the component time to failure is known precisely, then*

$$\underline{R}^*(t) = \max\left\{0, (m+1)F(t)-m\right\},\ \overline{R}^*(t) = \min\left(1, (n-m)F(t)\right).$$

Proof. The formulas follow directly from Theorem 2 and the condition $p_v = p_w = F(t)$. ■

This means, even though the probability distribution of the component time to failure is known precisely and the judgement of the component independence is not introduced, then only imprecise reliability can be found.

Corollary 5 *The lower system unreliability is defined only by the probability of the largest interval $[0,\alpha_w]$ which is included in $[0,t]$. The upper system unreliability is defined only by the probability of the smallest interval $[0,\alpha_v]$ which includes $[0,t]$.*

Proof. The proof is obvious. ■

Corollary 6 *There hold for the series system*

$$\underline{R}^*(t) = \underline{p}_w,\ \overline{R}^*(t) = \min\left(1, n\overline{p}_v\right),$$

and for the parallel system

$$\underline{R}^*(t) = \max\left\{0, n\underline{p}_w-(n-1)\right\},\ \overline{R}^*(t) = \overline{p}_v.$$

Proof. Similarly to the proof of Corollary 3. ■

Example 1 *Let us consider an 1-out-of-3 system consisting of identical components. Suppose that experts provided 5%, 50%, and 95% quantiles of an unknown probability distribution of the component time to failure: 5 days, 70 days, and 300 days. The assessments of the experts can be represented as follows:*

$$\Pr(X\leq 5) = 0.05,\ \Pr(X\leq 70) = 0.5,\ \Pr(X\leq 300) = 0.95.$$

Let us find bounds for the unreliability of the system at time 50 days, i.e., we have to find $\underline{R}(50)$ and $\overline{R}(50)$. By using notation introduced in this section, we can write

$$\begin{aligned}&\alpha_1 = 5,\ \alpha_2 = 70,\quad \alpha_3 = 300, \\ &p_1 = 0.05,\ p_2 = 0.5,\quad p_3 = 0.95,\ p_4 = 1.\end{aligned}$$

Then the values v and w are defined as

$$v = \min\{j : \alpha_j \geq 50\} = 2,\ w = \max\{j : \alpha_j \leq 50\} = 1.$$

1. *Components are independent. By using Theorem 1, we find* $\underline{R}(50)$ *and* $\overline{R}(50)$ *as follows:*

$$\underline{R}(50) = \sum_{i=2}^{3} \binom{3}{i} 0.05^i (1-0.05)^{3-i} = 0.00725,$$
$$\overline{R}(50) = \sum_{i=2}^{3} \binom{3}{i} 0.5^i (1-0.5)^{3-i} = 0.5.$$

2. *Lack of knowledge about independence of components. By using Theorem 2, we can find*

$$\underline{R}^*(50) = \max\{0, (2+1) \times 0.05 - 1\} = 0,$$
$$\overline{R}^*(50) = \min(1, (3-1) \times 0.5) = 1.$$

The values of $\underline{R}^*(50)$ *and* $\overline{R}^*(50)$ *show that it is impossible to make decision about reliability of the considered system because the initial information is too incomplete. If to add, for example, the judgement* $\Pr(X \leq 60) = 0.45$*, then we have the more precise bounds* $\underline{R}^*(50) = 0$*,* $\overline{R}^*(50) = 0.9$*.*

It should be noted that the assessments provided by the experts can be regarded as points of the exponential probability distribution of time to failure with the failure rate 0.01*. Let us find, for comparing results, the system reliability under the assumption that the precise exponential distribution of the component time to failure is available. By using Corollary 1, we get for independent components:*

$$\underline{R}(50) = \overline{R}(50) = \sum_{i=2}^{3} \binom{3}{i} (1 - \exp(-0.01 \cdot 50))^i (\exp(-0.01 \cdot 50))^{3-i}$$
$$= 0.3426.$$

By using Corollary 4, we have for the case of the lack of knowledge about independence of components:

$$\underline{R}^*(50) = \max\{0, (1+1)(1 - \exp(-0.01 \cdot 50)) - 1\} = 0,$$
$$\overline{R}^*(50) = \min(1, (3-1)(1 - \exp(-0.01 \cdot 50))) = 0.7869.$$

4. Probabilities on nested intervals

Consider a case with the following partial information about probabilities of failures:

$$\underline{p}_j \leq \Pr\{\underline{\alpha}_j \leq X_j \leq \overline{\alpha}_j\} \leq \overline{p}_j, \ j = 1, ..., l, \tag{12}$$

where

$$[\underline{\alpha}_1, \overline{\alpha}_1] \subset [\underline{\alpha}_2, \overline{\alpha}_2] \subset ... \subset [\underline{\alpha}_l, \overline{\alpha}_l]. \tag{13}$$

In other words, there are nested intervals $[\underline{\alpha}_j, \overline{\alpha}_j]$ with interval probabilities $[\underline{p}_j, \overline{p}_j]$ that the failure of a component is inside these intervals, respectively. Here we have to note the additional condition $\overline{\alpha}_{l+1} \to \infty$.

Let v be the maximal number j such that $\underline{\alpha}_j \geq t$, i.e. $v = \max\{j : \underline{\alpha}_j \geq t\}$. Let w be the maximal number j such that $\overline{\alpha}_j \leq t$, i.e. $w = \max\{j : \overline{\alpha}_j \leq t\}$.

4.1. Independent components

Theorem 3 *If the system components are statistically independent and governed by probabilities in the form of (12) and (13), then the lower and upper bounds for the unreliability at time t of the m-out-of-n system are computed as follows:*

$$\underline{R}(t)=\sum_{i=m+1}^{n}\binom{n}{i}\underline{p}_{w}^{i}(1-\underline{p}_{w})^{n-i},\ \overline{R}(t)=\sum_{i=m+1}^{n}\binom{n}{i}\underline{p}_{v}^{n-i}(1-\underline{p}_{v})^{i}.$$

Proof. Let us consider the 1-out-of-3 system for simplicity. First, we assume that $\underline{p}_i=\overline{p}_i=p_i$ for all $i=1,...,l$. Problems (10)-(11) differ from optimization problems for the case of nested intervals with the following constraints:

$$\sum_{i=1}^{l+1} I_{[\underline{\alpha}_k,\overline{\alpha}_k]}(x_i)c_i=p_k,\ k=1,...,l. \tag{14}$$

It can be shown (see Theorem 1) that values x_k delivering the optima meet the following conditions for all possible k: $x_k\in[\underline{\alpha}_k,\overline{\alpha}_k]\backslash[\underline{\alpha}_{k-1},\overline{\alpha}_{k-1}]$. The following proof coincides with the proof of Theorem 1, except the case of the upper bound. Therefore, we consider only this case. Since the system is in a working state if at least $n-m$ components are working, then at least $n-m$ of all different x_i must be larger than t, i.e. $i\leq v$. Then

$$\begin{aligned}\overline{R}&=1-\sum_{i=n-m}^{n}\binom{n}{i}\left(\sum_{k=1}^{v}c_k\right)^{i}\left(\sum_{k=v+1}^{l+1}c_k\right)^{n-i}\\&=\sum_{i=0}^{n-m-1}\binom{n}{n-i}p_v^{i}(1-p_v)^{n-i}=\sum_{i=m+1}^{n}\binom{n}{i}p_v^{n-i}(1-p_v)^{i}.\end{aligned}$$

The proof of the case when initial probabilities are interval-valued is similar to the proof of Theorem 1. ■

Corollary 7 *The lower system unreliability is defined only by the probability of the largest interval $[\underline{\alpha}_w,\overline{\alpha}_w]$ which is included in $[0,t]$. The upper system unreliability is defined only by the probability of the largest interval $[\underline{\alpha}_v,\overline{\alpha}_v]$ which intersects with $[0,t]$ is empty.*

Proof. Obviously from Theorem 3 and definitions of v and w. ■

Corollary 8 *There hold for the series system*

$$\underline{R}(t)=1-(1-\underline{p}_{w})^{n},\ \overline{R}(t)=1-\underline{p}_{v}^{n},$$

and for the parallel system

$$\underline{R}(t)=\underline{p}_{w}^{n},\ \overline{R}(t)=(1-\underline{p}_{v})^{n}.$$

Proof. Similarly to the proof of Corollary 3. ■

4.2. Lack of knowledge about independence of components

Theorem 4 *If the information about the m-out-of-n system components is given in the form of (12) and (13) and there is no information about independence of components, then there hold*

$$\underline{R}^{*}(t)=\max\left\{0,(m+1)\underline{p}_{w}-m\right\},\ \overline{R}^{*}(t)=\min\left(1,(n-m)(1-\underline{p}_{v})\right).$$

Proof. Let us consider the 1-out-of-3 system for simplicity. Let us introduce notation: D is the event $\{g(X_1, X_2, X_2) \leq t\}$, A_i is the event $\{X_i \in [\underline{\alpha}_i, \overline{\alpha}_i] \backslash [\underline{\alpha}_{i-1}, \overline{\alpha}_{i-1}]\}$ and A_i^c is the set complement to A_i, $A_i A_k$ is a subset of the universal set $A_{l+1} \times A_{l+1}$. The following proof is similar to the proof of Theorem 2, except for the computation of the probability $\max_{i,j,k:D^c \supset A_i A_j A_k} \underline{P}(A_i A_j A_k)$. Therefore, we consider only this case. By using the fact that the system is working if at least $n - m$ components are working, we can conclude that

$$\max_{i,j,k:D^c \supset A_i A_j A_k} \underline{P}(A_i A_j A_k) = \max \left\{ \max_{i \leq v, j \leq v, k \leq v} \underline{P}(A_i A_j A_k), \max_{i \leq v, j \leq v, k \geq 0} \underline{P}(A_i A_j A_k) \right\}$$
$$= \max \left(0,\ 2\underline{p}_v - 1 \right).$$

By generalizing the results on arbitrary m and n, we obtain

$$\max_{i,j,k:D^c \supset A_i \cdots A_k} \underline{P}(A_i \cdots A_k) = \max \left(0,\ (n-m)\underline{p}_v - (n-m-1) \right).$$

Hence

$$\overline{R}^* = \min \left\{ 1, 1 - (n-m)\underline{p}_v + (n-m-1) \right\}$$
$$= \min \left(1, (n-m)(1 - \underline{p}_v) \right),$$

as was to be proved. ■

Corollary 9 *If the information about the m-out-of-n system components is given as*

$$\underline{p} \leq \Pr\{\underline{\alpha} \leq X_i \leq \overline{\alpha}\} \leq \overline{p},$$

then there hold for $t = \underline{\alpha}$

$$\underline{R}^*(t) = 0,\ \overline{R}^*(t) = \min \left(1, (n-m)(1 - \underline{p}) \right),$$

and $t = \overline{\alpha}$

$$\underline{R}^*(t) = \max \left\{ 0, (m+1)\underline{p} - m \right\},\ \overline{R}^*(t) = 1.$$

Proof. This is obvious from the following. If $t = \underline{\alpha} \leq \overline{\alpha}$, then $v = \max\{j : \underline{\alpha}_j \geq t\} = 1$ and $w = \max\{j : \overline{\alpha}_j \leq t\} = 0$. If $t = \overline{\alpha}$, then $v = 0$ and $w = 1$. ■

Corollary 10 *The lower system unreliability is defined only by the probability of the largest interval* $[\underline{\alpha}_w, \overline{\alpha}_w]$ *which is included in* $[0, t]$*. The upper system unreliability is defined only by the probability of the largest interval* $[\underline{\alpha}_v, \overline{\alpha}_v]$ *which intersects with* $[0, t]$ *is empty.*

Proof. Obviously from Theorem 4 and definitions of v and w. ■

Corollary 11 *There hold for the series system*

$$\underline{R}^*(t) = \underline{p}_w,\ \overline{R}^*(t) = \min \left(1, n(1 - \underline{p}_v) \right).$$

and for the parallel system

$$\underline{R}^*(t) = \max \left\{ 0, n\underline{p}_w - (n-1) \right\},\ \overline{R}^*(t) = (1 - \underline{p}_v).$$

Proof. Similarly to the proof of Corollary 3. ■

It can be seen that the lower and upper bounds for the system unreliability depend only on the lower probabilities of the nested intervals. This implies that knowledge of upper probabilities does not give any useful information. Moreover, according to [16], the initial information can be considered as the possibility and necessity measures [17]. Indeed, according to [18], upper probabilities induced by a set of lower bounds $\{P(A_i) \geq \gamma_i, i = 1, ..., n\}$ are possibility measures if the set $\{A_1, ..., A_n\}$ is nested, that is, $A_1 \subset A_2 \subset ... \subset A_n$. The upper probability in this case coincides with the necessity $1 - \gamma_k$ of the event A_k. Denote

$$\pi(\underline{\alpha}_j) = \pi(\overline{\alpha}_j) = 1 - \underline{p}_j, \ j = 1, ..., m.$$

Then the time to failure X of components can be regarded as a fuzzy variable with the possibility distribution function $\pi(\underline{\alpha}_j) = \pi(\overline{\alpha}_j)$, $j = 1, ..., m$. Let us prove that the system reliability measure by such initial data can be also considered as the possibility and necessity measures of failure before time t.

Theorem 5 *If the initial information is represented as a set of probabilities defined on nested intervals, then either* $\underline{R}(t) = 0$ *or* $\overline{R}(t) = 1$.

Proof. It follows from the definition of v and w that if $v > 0$, then $w = 0$. In this case, there hold $\underline{p}_w = 0$ and $\underline{p}_v \geq 0$. By substituting these probabilities into the corresponding expressions for $\underline{R}(t)$, $\overline{R}(t)$, $\underline{R}^*(t)$, and $\overline{R}^*(t)$, we obtain $\underline{R}(t) = 0$, $\overline{R}(t) \geq 0$, $\underline{R}^*(t) = 0$, $\overline{R}^*(t) \geq 0$. If $w > 0$, then $v = 0$. In this case, there hold $\underline{p}_v = 0$ and $\underline{p}_w \geq 0$. By substituting these probabilities into the corresponding expressions for $\underline{R}(t)$, $\overline{R}(t)$, $\underline{R}^*(t)$, and $\overline{R}^*(t)$, we obtain $\underline{R}(t) \geq 0$, $\overline{R}(t) = 1$, $\underline{R}^*(t) \geq 0$, $\overline{R}^*(t) = 1$, as was to be proved. ■

Corollary 12 *If the initial information is represented as a set of probabilities defined on nested intervals, then* $\underline{R}(t)$ *and* $\overline{R}(t)$ *can be regarded as the possibility and necessity measures.*

Proof. This follows from Theorem 5 and the definition of possibility measures given in [16]. ■

By using the last result, the possibility distribution function of the system time to failure can be obtained as follows:

$$\pi_S(t) = \begin{cases} \overline{R}(t), & t \leq t_0 \\ 1, & t_0 \leq t \leq t_1 \\ 1 - \underline{R}(t), & t \geq t_1 \end{cases},$$

where $t_0 = \min\{t : \overline{R}(t) = 1\}$, or $t_1 = \max\{t : \underline{R}(t_0) = 0\}$.

The above reasoning allows us to obtain and to explain the reliability measure of the m-out-of-n system by fuzzy initial data. For example, there holds for the case of the lack of independence and by $t_0 = t_1$

$$\pi_S(t) = \begin{cases} \max\{0, (m+1)(1 - \pi(\underline{\alpha}_w)) - m\}, & t \leq t_0 \\ \min(1, (n-m)\pi(\underline{\alpha}_v)), & t \geq t_0 \end{cases}$$

$$= \begin{cases} 1 - \min\{1, (m+1)\pi(\underline{\alpha}_w))\}, & t \leq t_0 \\ \min(1, (n-m)\pi(\underline{\alpha}_v)), & t \geq t_0 \end{cases}.$$

5. Mean time to failure

Now we suppose that there is information only about the lower $\underline{a}$ and upper $\overline{a}$ MTTFs of components, i.e.

$$\underline{a} \leq \int_{\mathbb{R}_+} x\rho(x)\mathrm{d}x \leq \overline{a}.$$

Let us find the lower $\underline{T}$ and upper $\overline{T}$ MTTFs of the m-out-of-n system under condition that there is no information about the probability distribution functions of time to failure of components.

5.1. Independent components

Theorem 6 *Let the lifetime distribution $Q(t)$ of a system can be expressed through the lifetime distribution $q(t)$ of components as $Q(t) = G(q(t))$, where $G(x)$ is a non-negative continuous function defined on the interval $[0,1]$. We assume that there exists a finite limit $\lim_{x\to 0} G(x)/x$. Then the lower and upper MTTFs of the system are determined as*

$$\underline{T} = \underline{a} \min_{0\leq x\leq 1} \frac{G(x)}{x},\ \overline{T} = \overline{a} \max_{0\leq x\leq 1} \frac{G(x)}{x}.$$

If $x = 0$, then $G(x)/x = \lim_{x\to 0} G(x)/x$.

Proof. The MTTF of the system is determined as

$$T = \int_{\mathbb{R}_+} Q(t)\mathrm{d}t = \int_{\mathbb{R}_+} G(q(t))\mathrm{d}t.$$

Since $G(0) = 0$, then we assume that the integration variable takes the values for which $q(t) > 0$. Then

$$T = \int_{\mathbb{R}_+} q(t)\frac{G(q(t))}{q(t)}\mathrm{d}t.$$

This implies that

$$\underline{a} \min_{0\leq x\leq 1} \frac{G(x)}{x} \leq T \leq \overline{a} \max_{0\leq x\leq 1} \frac{G(x)}{x}.$$

Let us show that we have obtained the sharp bounds. Let $x_{\min}$ and $x_{\max}$ be points such that

$$\min_{0\leq x\leq 1} \frac{G(x)}{x} = \frac{G(x_{\min})}{x_{\min}},\ \max_{0\leq x\leq 1} \frac{G(x)}{x} = \frac{G(x_{\max})}{x_{\max}}.$$

Suppose

$$q_{\min}(t) = \begin{cases} 1, & t = 0 \\ x_{\min}, & 0 \leq t \leq \underline{a}/x_{\min} \\ 0 & t > \underline{a}/x_{\min} \end{cases},\ q_{\max}(t) = \begin{cases} 1, & t = 0 \\ x_{\max}, & 0 \leq t \leq \overline{a}/x_{\max} \\ 0 & t > \overline{a}/x_{\max} \end{cases}.$$

For these functions, we have

$$\underline{T} = \int_{\mathbb{R}_+} G(q_{\min}(t))\mathrm{d}t = G(x_{\min})\frac{\underline{a}}{x_{\min}} = \underline{a} \min_{0\leq x\leq 1} \frac{G(x)}{x},$$
$$\overline{T} = \int_{\mathbb{R}_+} G(q_{\max}(t))\mathrm{d}t = G(x_{\max})\frac{\overline{a}}{x_{\max}} = \overline{a} \max_{0\leq x\leq 1} \frac{G(x)}{x}.$$

This completes the proof. ■

Theorem 7 *The lower and upper MTTFs of the m-out-of-n system with independent identical components are determined as follows:*

$$\underline{T} = \begin{cases} 0, & \text{if } m+1 < n, \\ \underline{a}, & \text{if } m+1 = n, \end{cases},\ \overline{T} = \overline{a}\binom{n}{m}(n-m)\frac{t_0^{n-m-1}}{(1+t_0)^{n-1}},$$

where t_0 is defined from the following equation:

$$\sum_{k=0}^{m} \binom{n}{k} t_0^{m-k} = \binom{n}{m}(n-m).$$

Proof. There holds for the m-out-of-n system

$$G(x)=\sum_{k=0}^{m}\binom{n}{k}(1-x)^{k}x^{n-k}.$$

Therefore, the minimum and maximum of the function

$$y=\frac{G(x)}{x}=\sum_{k=0}^{m}\binom{n}{k}(1-x)^{k}x^{n-k-1}$$

have to be found. Let $m<n-1$. If $x=0$, then $y=0$ and $\underline{T}=0$. If $x=1$, then $y=1$. In this case, we obtain

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}&=-\sum_{k=1}^{m}\binom{n}{k}k(1-x)^{k-1}x^{n-k-1}+\sum_{k=0}^{m}\binom{n}{k}(n-k-1)(1-x)^{k}x^{n-k-2}\\&=-\sum_{k=0}^{m-1}\binom{n}{k}(1-x)^{k}x^{n-k-2}+\binom{n}{m}(n-m-1)(1-x)^{m}x^{n-m-2}.\end{aligned}$$

So, we have the following equation

$$\sum_{k=0}^{m-1}\binom{n}{k}(1-x)^{k}x^{n-k-2}=\binom{n}{m}(n-m-1)(1-x)^{m}x^{n-m-2}$$

or

$$\sum_{k=0}^{m}\binom{n}{k}\left(\frac{x}{1-x}\right)^{m-k}=\binom{n}{m}(n-m).$$

Denote $t=x/(1-x)$ $(0<t<\infty)$. Then the equation

$$\sum_{k=0}^{m}\binom{n}{k}t^{m-k}=\binom{n}{m}(n-m)$$

has one solution $t_0>0$. This implies that $x_0=t_0/(1+t_0)$ is the maximum of y and

$$\begin{aligned}\overline{T}&=\overline{a}\sum_{k=0}^{m}\binom{n}{k}(1-x_0)^{k}x_0^{n-k-1}=\overline{a}\sum_{k=0}^{m}\binom{n}{k}\frac{t_0^{n-k-1}}{(1+t_0)^{n-1}}\\&=\overline{a}\frac{t_0^{n-m-1}}{(1+t_0)^{n-1}}\sum_{k=0}^{m}\binom{n}{k}t_0^{m-k}=\overline{a}\binom{n}{m}(n-m)\frac{t_0^{n-m-1}}{(1+t_0)^{n-1}}.\end{aligned}$$

■

Let us consider a special case $m=1$. In this case, $t+n=n(n-1)$ and $t_0=n(n-2)$. Then

$$\overline{T}=\overline{a}n(n-1)\frac{(n(n-2))^{n-2}}{(1+n(n-2))^{n-1}}=\overline{a}\frac{n}{n-1}\left(\frac{n(n-2)}{(n-1)^2}\right)^{n-2}.$$

5.2. Lack of information about independence of components

Theorem 8 *The lower and upper MTTFs of the m-out-of-n system are determined as follows:*

$$\underline{T}=\begin{cases}0, & \text{if } m+1<n,\\ \max_{1\leq i\leq n}\underline{a}_i, & \text{if } m+1=n,\end{cases},\ \overline{T}=\min_{\substack{i_1<i_2<\ldots<i_k\\ m+1\leq k\leq n}}\frac{\sum_{j=1}^{k}\overline{a}_{i_j}}{k-m},$$

where $\underline{a}_i$ and $\overline{a}_i$ are the lower and upper MTTFs of the i-th component.

The proof of the theorem can be found in [13].

Corollary 13 *The lower and upper MTTFs of a m-out-of-n system consisting of identical components are determined as follows:*

$$\underline{T} = \begin{cases} 0, & \text{if } m+1 < n, \\ \underline{a}, & \text{if } m+1 = n, \end{cases}, \quad \overline{T} = \overline{a}\frac{n}{n-m}.$$

Proof. This follows from Theorem 8. ■

6. Conclusion

It should be noted that only m-out-of-n systems with identical components has been investigated in the paper. However, the proposed expressions can be easily extended on a case of different components. Moreover, the obtained results can be successfully used in more complex reliability calculations by means of a method proposed in [19]. This method allows us to calculate the system reliability by an arbitrary number of constraints in optimization problems (1)-(2) and (3)-(4) under condition that the explicit expressions for systems consisting of identical components are available. Therefore, this paper can be considered as a basis for computing the reliability of m-out-of-n systems under different types of partial initial information.

Acknowledgement

I am very grateful to Prof. Dr. Kurt Weichselberger (Munich University, Germany) for his scientific help and support. I also thank the anonymous referees for useful suggestions that improved the paper.

Appendix

Theorem 9 *If an optimal solution of optimization problems (1)-(2) exists, then it can be found among the following densities*

$$\rho^*(\mathbf{X}) = \sum_{k=1}^{N+1} c_k \delta_{\mathbf{X}_k}(\mathbf{X}), \ N = \sum_{i=1}^{n} m_i \tag{15}$$

where $\mathbf{X}_k = (x_1^{(k)}, ..., x_n^{(k)}) \in \mathbb{R}_+^n$, $c_k \in \mathbb{R}_+$, $\delta_{\mathbf{X}_k}(\mathbf{X})$ *is the Dirac function having unit area concentrated in the immediate vicinity of points* $\mathbf{X}_k$.

By substituting (15) into (1)-(2), we obtain

$$\underline{\mathbb{E}}(g) = \inf_{c_k, \mathbf{X}_k} \sum_{k=1}^{N+1} c_k g(\mathbf{X}_k), \ \overline{\mathbb{E}}(g) = \sup_{c_k, \mathbf{X}_k} \sum_{k=1}^{N+1} c_k g(\mathbf{X}_k),$$

subject to

$$\sum_{k=1}^{N+1} c_k = 1, \ c_k \geq 0, \ k = 1, ..., N+1,$$

$$\underline{a}_{ij} \leq \sum_{k=1}^{N+1} c_k \varphi_{ij}(x_i^{(k)}) \leq \overline{a}_{ij}, \ i \leq n, \ j \leq m_i.$$

Theorem 10 *If an optimal solution of optimization problem (3)-(4) exists, then it can be found among the following densities*

$$\rho_k^*(x) = \sum_{j=1}^{m_k+1} c_j^{(k)} \delta_{x_k^{(j)}}(x), \ k = 1, ..., n, \tag{16}$$

where $x_k^{(j)} \in \mathbb{R}_+$, $c_j^{(k)} \in \mathbb{R}_+$.

By substituting (16) into (3)-(4), we obtain

$$\underline{\mathbb{E}}(g) = \inf_{c_j, \mathbf{X}_j} \sum_{l_1=1}^{m_1+1} ... \sum_{l_n=1}^{m_n+1} g(x_1^{(l_1)}, ..., x_n^{(l_n)}) \prod_{v=1}^{n} c_{l_v}^{(v)},$$

subject to

$$\sum_{k=1}^{m_l+1} c_k^{(l)} = 1, \ c_k^{(l)} \geq 0, \ l = 1, ..., n,$$

$$\underline{a}_{ij} \leq \sum_{l=1}^{m_i+1} \varphi_{ij}(x_i^{(l)}) c_l^{(i)} \leq \overline{a}_{ij}, \ i \leq n, \ j \leq m_i.$$

The proofs of Theorems 9 and 10 are based on two lemmas.

Lemma 1 *Suppose that functions* $g^{(i)}(t)$, $i = 1, ..., m$, *and* $g(t)$ *are integrable on* $[0, \infty)$. *Then an optimal solution of the problem*

$$z = \max_{\Phi} \int_{\mathbb{R}_+} g(t) H(t) \mathrm{d}t,$$

subject to

$$\int_{\mathbb{R}_+} g^{(i)}(t) H(t) \mathrm{d}t = a_i, \ i = 1, ..., m,$$

can be found in a class of distributions focusing on $m+1$ *points. Here* $H(t)$ *is a non-increasing function such that* $H(t) \geq 0$, $H(0) = 1$; Φ *is a set of all possible functions* $H(t)$ *satisfying the constraints.*

Proof. Let us consider a discrete optimization problem:

$$z = \max \sum_{k=0}^{n} g_k x_k,$$

subject to

$$\sum_{k=0}^{n} g_k^{(i)} x_k = a_i, \ i = 1, ..., m, \ 1 = x_0 \geq x_1 \geq ... \geq x_n \geq 0.$$

Let us introduce new variables $\alpha_k = x_k - x_{k+1}$, $k = 0, 1, 2, ..., n-1$, $\alpha_n = x_n$. Hence $x_k = \alpha_k + \alpha_{k+1} + ... + \alpha_n$. Now the following equivalent problem can be written:

$$z = \max \sum_{j=0}^{n} \sum_{k=0}^{j} g_k \alpha_j,$$

subject to

$$\sum_{j=0}^{n}\sum_{k=0}^{j} g_k^{(i)}\alpha_j = a_i,\ i=1,..,m,\ \alpha_k \geq 0,\ k=0,..,n.$$

This is a linear optimization problem in the canonical form. The constraints are the system of linear equations of dimension $m \times (n+1)$. It is known that an optimal solution of such the problem can be found among the basic solutions for which only m components are non-zero. Suppose that the non-zero components are $\alpha_{k_1}, ..., \alpha_{k_m}$. Then the following equalities hold:

$$\begin{aligned} x_0 &= x_1 = ... = x_{k_1}, \\ x_{k_1+1} &= x_{k_1+2} = ... = x_{k_2}, \\ &... \\ x_{k_m+1} &= x_{k_m+2} = ... = x_n. \end{aligned}$$

This implies that there are m jumps in the sequence $\{x_k\}$. If the last term of the sequence is $x_n > 0$, then there is the $(m+1)$-th jump. Note that m does not dependent on n. Then the passage to the limit as $n \to \infty$ completes the proof. ■

Lemma 2 *Suppose that functions $g^{(i)}(t)$, $i = 1, ..., m$, and $g(t)$ are integrable on $[0, \infty)$. Then an optimal solution of the problem*

$$z = \max_{\Phi} \int_{\mathbb{R}_+} g(t)H(t)\mathrm{d}t,$$

subject to

$$\underline{a}_i \leq \int_{\mathbb{R}_+} g^{(i)}(t)H(t)\mathrm{d}t \leq \overline{a}_i,\ i=1,...,m,$$

can be found in a class of distributions focusing on $m+1$ points. Here $H(t)$ is a non-increasing function such that $H(t) \geq 0$, $H(0) = 1$; Φ is a set of all possible functions $H(t)$ satisfying constraints.

Proof. We write a discrete optimization problem

$$z = \max \sum_{j=0}^{n}\sum_{k=0}^{j} g_k\alpha_k,$$

subject to

$$\underline{a}_i \leq \sum_{j=0}^{n}\sum_{k=0}^{j} g_k^{(i)}\alpha_k \leq \overline{a}_i,\ i=1,...,m,\ \alpha_k \geq 0,\ k=0,1,...,n,$$

as it has been done in the proof for Lemma 1. Let us rewrite m constraints in the matrix form $\underline{A} \leq G \cdot X \leq \overline{A}$, where G is a matrix with components $g_k^{(i)}$; X, $\underline{A}$ and $\overline{A}$ are vectors with components α_k, $\underline{a}_i$ and $\overline{a}_i$, respectively. Let us fix such a vector Y that $\underline{A} \leq Y \leq \overline{A}$. Then an optimization problem with constraints $G \cdot X = Y$ satisfies Lemma 1, i.e. the optimal solution has m non-zero components. Let the initial optimization problem has an optimal solution X^*. Then there exists $Y^* = GX^*$ such that $\underline{A} \leq Y^* \leq \overline{A}$. However, Lemma 1 implies that the fixed vector Y^* has m non-zero components. This completes the proof. ■

Proof of Theorem 9. If we replace the variable t by the vector $\mathbf{X}$, then the condition of Lemma 2 is not changed and the optimal solution to the problem

$$z = \max_{\Phi} \int_{\mathbb{R}_+^n} g(\mathbf{X}) H(\mathbf{X}) \mathrm{d}\mathbf{X},$$

subject to

$$\underline{a}_i \leq \int_{\mathbb{R}_+^n} g^{(i)}(\mathbf{X}) H(\mathbf{X}) \mathrm{d}\mathbf{X} \leq \overline{a}_i,\ i \leq N,$$

is a n-dimensional distribution function $H^*(\mathbf{X})$ having $N+1$ jumps at points $\mathbf{X}_k = (x_1^{(k)}, ..., x_n^{(k)})$, $k \leq N$. Then

$$\rho^*(\mathbf{X}) = \frac{\partial^n H^*(\mathbf{X})}{\partial x_1 \cdots \partial x_n} = \sum_{k=1}^{N+1} c_k \delta_{\mathbf{X}_k}(\mathbf{X}),$$

where c_k is a length of the k-th jump. ∎

Proof of Theorem 10. Let an optimal solution to the initial problem be $\rho_1^*(x_1) \cdots \rho_n^*(x_n)$. Let us fix all ρ_i^*, $i \neq k$, except ρ_k, $k \in \{1, ..., n\}$. We obtain an optimal solution of the problem because the number of constraints for the problem of one unknown density ρ_k is m_k. ∎

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