Imprecise reliability for some new lifetime distribution classes

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Abstract

To develop a general reliability theory, taking into account various sources of information and a lack of satisfactory data on which estimates of system parameters can be based, the theory of coherent imprecise probabilities can be used. The purpose of the paper is to study the reliability of systems based on coherent imprecise probability models, taking into account the ageing aspect of the lifetime distributions, independence of system components, and a lack of satisfactory data. We use some new non-parametric life distribution classes which generalize the well-known increasing and decreasing failure rate distributions and can represent various judgements related to lifetime distributions. In this paper we apply the theory of coherent imprecise probabilities to reliability analysis of monotone systems.

Keywords: Imprecise probabilities, natural extension, monotone systems, mean time to failure.

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1 Introduction

Reliability assessments that are combined to describe systems and components may come from various sources. Some may be objective measures, based on relative frequencies or on well established statistical models. Some of the reliability assessments may be supplied by experts or engineers. Especially in practical reliability problems, the use of judgements of engineers may be important, since they may be the only source of information. The reliability assessments may be conveyed by statements in natural language, because natural language expressions are often more appropriate for expressing reliability information than numerical expressions. To develop a general reliability theory, taking into account various sources of information and a lack of satisfactory data on which estimates of system parameters can be based, the theory of coherent imprecise probabilities (Walley, 1991; Walley, 1996; Kuznetsov, 1991; Kuznetsov, 1995) can be used. A general framework for the theory of coherent imprecise probabilities is provided by upper and lower previsions. According to de Cooman (1996) and Walley (1996), they can model a very wide variety of kinds of uncertainty, partial information, and ignorance. Walley's theory of imprecise probabilities is arguably the most satisfactory of all current theories of uncertain reasoning from a foundational point of view (Moral and Wilson, 1995).

Coolen (1996), Coolen and Newby (1994) have shown how some commonly used concepts in reliability theory can be extended in a sensible way and combined with prior knowledge through the use of coherent imprecise probabilities. However, they provide parametric models for lifetimes.

Suppose that the information we have about the functioning of components and systems is conveyed by imprecise statements. For example, judgements of engineers may have the form "$MTTF$ (mean time to failure) of component A equals 10 hours and MTTF of component B is between 3 and 5 hours". How can we compute the reliability of a series system consisting of these components from such partial information? If we do not know the component lifetime distribution, then the problem cannot be solved by means of classical reliability methods. An alternative method is to consider the available reliability measures as lower and upper previsions, and to use the theory of imprecise probabilities for calculating the system reliability measures as new previsions. In particular, the problem stated in the above example has been solved by means of a general procedure called the natural extension (Utkin, 1998; Utkin and Gurov, 1998; Utkin and Gurov, 1999), which produces a coherent overall model from a collection of judgements and can be regarded as a linear optimization problem (Walley, 1991; Kuznetsov, 1991).

At the same time, there are judgements whose representation by lower and upper previsions is a difficult problem. Suppose that we obtain some additional information such as "long infancy and wear-out periods for component A were observed, only the wear-out period for component B was observed, and the components are independent". These judgements take into account the ageing aspect of the lifetime distributions (three phases of the so-called "bathtub" curve characterizing the lifetime evolution of a system: early failure, useful life, wear-out periods) and a condition of independence of components. Now the constraints in the optimization problem are non-linear and the computation of the natural extension is more complicated. How can we compute the reliability of the system from such additional information?

The purpose of the paper is to study the reliability of systems based on coherent imprecise probability models, taking into account the ageing aspect of the lifetime distributions, independence of system components, and a lack of satisfactory data. We introduce some new non-parametric life distribution classes which generalize the well-known increasing and decreasing failure rate distributions (Barlow, Marshall, and Proschan, 1963; Barlow and Proschan, 1975) and can represent various judgements related to lifetime distributions. In this paper we apply the theory of coherent imprecise probabilities to reliability analysis of unrepairable systems.

The paper is organized as follows. In Section 2, a general approach to reliability analysis of systems on the basis of imprecise probability theory is considered. Some new distribution classes and their properties are introduced in Section 3. The preliminary results required for reliability analysis of series and parallel systems are proved in Section 4. Reliability analysis of series, parallel, and monotone systems, using some new lifetime distribution classes, is given in Section 5. In Section 6, similar results are studied for discrete lifetime distributions.

2 Natural extension

Consider a system consisting of n components. Let $f_{ij}(x_i)$ be a function of the i-th component lifetime x_i, $j = 1, ..., m_i$. Here m_i is the number of quantitative or qualitative judgements that are related to the i-th component. According to Barlow and Proschan (1975), the system lifetime is uniquely determined by the component lifetimes. Denote $\mathbf{X} = (x_1, ..., x_n)$. Then there exists a function $g(\mathbf{X})$ of the component lifetimes that characterizes the system reliability. A *gamble* is a real-valued function on a possibility space whose value is uncertain (Walley, 1991; Kuznetsov, 1991). Then the functions $f_{ij}(x_i)$ and $g(\mathbf{X})$ can be regarded as gambles. Suppose that partial statistical information is represented as a set of lower and upper previsions $\underline{a}_{ij} = \underline{M}(f_{ij}(x_i))$, $\overline{a}_{ij} = \overline{M}(f_{ij}(x_i))$, $i = 1, ..., n$, $j = 1, ..., m_i$.

For example, consider a series system consisting of two components. Suppose that we know only upper bounds μ_1, μ_2 for two moments of the first component lifetime and the upper probability p that the second component fails in the interval $[0, \tau]$. Then we have the following set of previsions: $\overline{M}(x_1) = \mu_1$, $\overline{M}(x_1^2) = \mu_2$, $\overline{M}(I_{[0,\tau]}(x_2)) = p$. Here $I_{[0,\tau]}(x_2) = 1$ if $x_2 \in [0, \tau]$, $I_{[0,\tau]}(x_2) = 0$ if $x_2 \notin [0, \tau]$. If we need to find bounds for the first moment of the system lifetime (MTTF), then $g(x_1, x_2) = \min(x_1, x_2)$ and the system MTTFs can be regarded as lower and upper previsions $\underline{M}(g)$ and $\overline{M}(g)$.

For computing new lower and upper previsions $\overline{M}(g)$ and $\underline{M}(g)$ that characterize the system reliability, the natural extension can be used in the following form (Utkin, 1998; Utkin and

Gurov, 1998; Utkin and Gurov, 1999):

$$\begin{aligned}\overline{M}(g) &= \min_{c,c_{ij},d_{ij}} \left(c + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \left(c_{ij}\overline{a}_{ij} - d_{ij}\underline{a}_{ij} \right) \right), \\ \underline{M}(g) &= -\overline{M}(-g),\end{aligned} \tag{1}$$

subject to $c_{ij} \in \mathbf{R}^+$, $d_{ij} \in \mathbf{R}^+$, $c \in \mathbf{R}$, and

$$g(\mathbf{X}) \leq c + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \left(c_{ij} f_{ij}(x_i) - d_{ij} f_{ij}(x_i) \right), \ \forall x_i \geq 0.$$

Returning to the above example, we can write:

$$\begin{aligned}\overline{M}(g) &= \min_{c,c_{11},c_{12},d_{21}} (c + c_{11}\mu_1 + c_{12}\mu_2 + c_{21}p), \\ \underline{M}(g) &= -\overline{M}(-g),\end{aligned}$$

subject to $c_{11}, c_{12}, c_{21} \in \mathbf{R}^+$, $c \in \mathbf{R}$, and $\forall x_i \geq 0$,

$$\min(x_1, x_2) \leq c + c_{11}x_1 + c_{12}x_1^2 + c_{21}I_{[0,\tau]}(x_2).$$

So, new lower and upper previsions $\overline{M}(g)$ and $\underline{M}(g)$ can be computed as a solution to a linear programming problem. Generally, the above programming problem may involve infinitely many constraints. However, their number can be greatly reduced for a lot of special cases (Utkin and Gurov, 1999; Utkin and Gurov, 1999a).

The natural extension in the form of the linear optimization problem is a powerful tool. However, it has some limitations. For instance, independence relationships cannot be represented simply in terms of gambles, since they are non-linear. The same difficulties arise when there is additional information about the probability distributions or distribution classes of the component lifetimes. In this case, the natural extension can be written in the form of expectations (Walley, 1991; Kuznetsov, 1995):

$$\begin{aligned}\overline{M}(g) &= \max \int_0^\infty \cdots \int_0^\infty g(\mathbf{X})\rho(\mathbf{X})\mathrm{d}\mathbf{X}, \\ \underline{M}(g) &= \min \int_0^\infty \cdots \int_0^\infty g(\mathbf{X})\rho(\mathbf{X})\mathrm{d}\mathbf{X},\end{aligned} \tag{2}$$

subject to

$$\begin{aligned}
\int_0^\infty \cdots \int_0^\infty \rho(\mathbf{X})\mathrm{d}\mathbf{X} &= 1,\ \rho(\mathbf{X}) \geq 0, \\
\int_0^\infty \cdots \int_0^\infty f_{ij}(x_i)\rho(\mathbf{X})\mathrm{d}\mathbf{X} &\in [\underline{a}_{ij}, \overline{a}_{ij}],\ i \leq n,\ j \leq m_i.
\end{aligned}$$

If components are independent, then $\rho(\mathbf{X}) = \rho(x_1)\cdots\rho(x_n)$. It should be noted that there are different mathematical definitions of independence (Walley, 1991). Here we consider the definition of independence in the sense of classical probability theory. Returning to the above example under the condition of independence, we can write:

$$\begin{aligned}
\overline{M}(g) = \max \quad & \int_0^\infty \int_0^\infty \min(x_1, x_2)\rho_1(x_1)\rho_2(x_2)\mathrm{d}x_1\mathrm{d}x_2, \\
\underline{M}(g) = \min \quad & \int_0^\infty \int_0^\infty \min(x_1, x_2)\rho_1(x_1)\rho_2(x_2)\mathrm{d}x_1\mathrm{d}x_2,
\end{aligned}$$

subject to

$$\begin{aligned}
\int_0^\infty \rho_i(x)\mathrm{d}x &= 1,\ \rho_i(x) \geq 0,\ i = 1, 2, \\
\int_0^\infty x\rho_1(x)\mathrm{d}x &\leq \mu_1,\ \int_0^\infty x^2\rho_1(x)\mathrm{d}x \leq \mu_2,\ \int_0^\tau \rho_2(x)\mathrm{d}x \leq p.
\end{aligned}$$

Note that the integral $\int_0^\infty \cdots \int_0^\infty g(\mathbf{X})\rho(\mathbf{X})\mathrm{d}\mathbf{X}$ can be represented in a different form. Let $Y = g(\mathbf{X})$ and $H(y) = \Pr(Y \geq y)$. Then

$$\int_0^\infty \cdots \int_0^\infty g(\mathbf{X})\rho(\mathbf{X})\mathrm{d}\mathbf{X} = \int_0^\infty H(y)\mathrm{d}y.$$

3 Distribution classes

In order to formalize judgements about the ageing aspects of lifetime distributions, we introduce some new flexible distribution classes and briefly investigate their properties. The probability distribution of the component (or system) lifetime X can be written as $H(t) = \Pr(X \geq t) = \exp(-\Lambda(t))$, where $\Lambda(t) = \int_0^t \lambda(x)\mathrm{d}x$ and $\lambda(t)$ is the time-dependent failure rate. Obviously, the function Λ is non-decreasing and $\Lambda(0) = 0$. Let r and s be numbers such that $0 \leq r \leq s \leq +\infty$.

Let us define a distribution class $\mathcal{H}(r,s)$ as follows. A probability distribution belongs to $\mathcal{H}(r,s)$ if $\Lambda(t)/t^r$ increases and $\Lambda(t)/t^s$ decreases as t increases. In that case we will write $\Lambda \in \Lambda(r,s) = \{\Lambda : \exp(-\Lambda) \in \mathcal{H}(r,s)\}$.

Next we list several special cases of the distribution classes $\mathcal{H}(r,s)$:

1. $\mathcal{H}(1,+\infty)$ is the class of all *IFRA* (increasing failure rate average) distributions (Barlow and Proschan, 1975);

2. $\mathcal{H}(r,s)$, $1 \leq r < s$, is the class of all IFRA distributions whose failure rate has a bounded rate of increase limited by minimal r and maximal s indices;

3. $\mathcal{H}(0,1)$ is the class of all *DFRA* (decreasing failure rate average) distributions (Barlow and Proschan, 1975);

4. $\mathcal{H}(r,s)$, $r < s \leq 1$, is the class of all DFRA distributions whose failure rate has a bounded rate of decrease limited by minimal r and maximal s indices;

5. $\mathcal{H}(r,s)$, $r < 1 < s$, is a class of distributions whose failure rate is non-monotone. Note that these distributions are the most popular in reliability because they characterize periods of component wear-in and wear-out.

Let us state some properties of distributions from $\mathcal{H}(r,s)$ without proofs:

1. If $r_2 \leq r_1 \leq s_1 \leq s_2$, then $\Lambda(r_1,s_1) \subset \Lambda(r_2,s_2)$ and $\mathcal{H}(r_1,s_1) \subset \mathcal{H}(r_2,s_2)$.

2. The function $-\ln(H(t))/t^s$ is decreasing in t, and the function $-\ln(H(t))/t^r$ is increasing in t.

3. The following inequalities hold: $H(t)^{\alpha^r} \leq H(\alpha t) \leq H(t)^{\alpha^s}$, $0 < \alpha < 1$.

4. The following inequalities hold: $r\Lambda(t)/t \leq \lambda(t) \leq s\Lambda(t)/t$.

5. The Weibull distribution with the arbitrary scale parameter λ and shape parameter q belongs to $\mathcal{H}(q,q)$.

6. If $\rho(t)$ is the density function of a random variable X and $H(t) = \Pr(X \geq t)$, let
$$r_0 = \min_t \frac{-t\rho(t)}{H(t)\ln(H(t))}, \quad s_0 = \max_t \frac{-t\rho(t)}{H(t)\ln(H(t))}.$$
Then for any values $r > r_0$ and $s < s_0$, $H(t) \notin \mathcal{H}(r,s)$. For any values $r \leq r_0$ and $s \geq s_0$, $H(t) \in \mathcal{H}(r,s)$.

7. The Gamma distribution with the density function $\rho_k(t) = \lambda^k t^{k-1} e^{-\lambda t}/\Gamma(k)$ belongs to $\mathcal{H}(1,k)$ for $k \geq 1$ and to $\mathcal{H}(k,1)$ for $k < 1$. Here Γ is the gamma function.

8. Let
$$\lambda(t) = \begin{cases} c_1 - d_1 t, & 0 \leq t \leq A \\ c_2, & A \leq t < B \\ c_3 + d_3 t, & t \geq B \end{cases}.$$
be the failure rate of $H(t)$ and c_1, c_2, c_3, d_1, $d_2 \in \mathbf{R}^+$. Here $A \leq c_1/d_1$. Then $H(t) \in \mathcal{H}(r,2)$, where
$$r = \frac{c_1 - d_1 A}{c_1 - d_1 A/2}.$$

Example 1 *Let λ be a non-decreasing function. Then Λ is convex and $n\Lambda(t) \leq \Lambda(nt)$, $n \geq 1$. This implies that the function $\Lambda(t)/t$ is increasing and $\Lambda \in \Lambda(1,+\infty)$.*

Example 2 *Let λ be a non-increasing function. Then Λ is concave and $n\Lambda(t) \geq \Lambda(nt)$, $n \geq 1$. This implies that the function $\Lambda(t)/t$ is decreasing and $\Lambda \in \Lambda(0,1)$.*

Example 3 *Let*
$$\lambda(t) = \begin{cases} 2 - t, & 0 \leq t \leq 1 \\ 1, & 1 < t < 2 \\ t, & t \geq 2 \end{cases}.$$
It follows from Property 8 that $\Lambda \in \Lambda(2/3,2)$.

4 Basic lemmas

In this section we introduce some preliminary results. When we have additional information about lifetime distributions, problem (2) can be rewritten as follows:

$$\overline{M}(g) = \max \int_0^\infty \Pr(g(\mathbf{X}) > t)\mathrm{d}t,\ \underline{M}(g) = \min \int_0^\infty \Pr(g(\mathbf{X}) > t)\mathrm{d}t, \tag{3}$$

subject to

$$\Pr(f_{ij}(x_i) > t) \in \mathcal{H}(r_i, s_i),$$
$$\int_0^\infty \Pr(f_{ij}(x_i) > t)\mathrm{d}t \in [\underline{a}_{ij}, \overline{a}_{ij}],\ i \leq n,\ j \leq m_i.$$

A solution of (3) can be easily obtained if we find a way to represent $\Pr(g(\mathbf{X}) > t)$ as $c(t)\Pr(f_{ij}(x_i) > t)$, where c is a monotone function. If $f_{ij}(x_i) = x_i$ and $g(\mathbf{X}) = x$, then the corresponding lower and upper previsions can be regarded as lower and upper MTTFs of the i-th component and the system, respectively. In this case, it can be shown that $\Pr(g(\mathbf{X}) > t) = c(t)\Pr(f_{ij}(x_i) > t)$ and the function c is monotone for series and parallel systems.

For example, if $g(x_1, x_2) = \min(x_1, x_2)$, then

$$\Pr(\min(x_1, x_2) > t) = \Pr(x_1 > t)\Pr(x_2 > t) = c(t)\Pr(x_2 > t),$$

where $c(t) = \Pr(x_1 > t)$.

Therefore, we need to prove the following Lemmas which give a way to solve problem (3) in several important special cases.

Lemma 1 *Suppose $b_k \geq 0$, $k = 1, ..., n$, is a monotonically increasing sequence. Denote*

$$z(\alpha_1, ..., \alpha_n) = \frac{\sum_{k=1}^n c_k \exp\left(-b_k(\alpha_1 + ... + \alpha_k)\right)}{\sum_{k=1}^n \exp\left(-b_k(\alpha_1 + ... + \alpha_k)\right)}.$$

If the sequence $c_1, ..., c_n$ is increasing, then the function z is monotonically decreasing in each variable $\alpha_k \geq 0$, $k = 1, ..., n$, with the other variables held fixed. If the sequence $c_1, ..., c_n$ is decreasing, then the function z is monotonically increasing in each α_k.

Proof. Denote $B_k = \exp\left(-b_k(\alpha_1 + ... + \alpha_k)\right)$. The derivative of z with respect to α_j is of the form:

$$\begin{aligned}\frac{\partial z}{\partial \alpha_j} &= \frac{\sum_{k,l=j,k>l}^{n}(c_l - c_k)(b_k - b_l)(B_k + B_l)}{\left(\sum_{k=1}^{n} B_k\right)^2} \\ &+ \frac{\sum_{k=j}^{n}\sum_{l=1}^{j-1}(c_l - c_k)b_k(B_k + B_l)}{\left(\sum_{k=1}^{n} B_k\right)^2}.\end{aligned}$$

Since $b_k - b_l \geq 0$ for $k > l$, then for the increasing sequence c_k, the inequality $\frac{\partial z}{\partial \alpha_j} \leq 0$ holds. This implies that z is decreasing in α_j. For the decreasing sequence c_k, the inequality $\frac{\partial z}{\partial \alpha_j} \geq 0$ holds. This implies that z is increasing in α_j. $\square$

Lemma 2 *Let c be a monotone function. Then the following optimization problem*

$$z = \max \int_a^b c(t) \exp(-\Lambda(t)) \mathrm{d}t$$

subject to $\Lambda \in \Lambda(r, s)$, $\int_a^b \exp(-\Lambda(t))\mathrm{d}t = d$, has a solution. If c is decreasing, then z achieves its maximum at $\Lambda(t) = qt^s$. If c is increasing, then z achieves its maximum at $\Lambda(t) = qt^r$. Here q is a constant.

Proof. Let us consider the following optimization problem:

$$z = \max \left(\alpha \sum_{k=1}^{n} c_k \exp\left(-x_k\right) \right)$$

subject to $\alpha \sum_{k=1}^{n} \exp\left(-x_k\right) = d$. Here $\alpha = (b - a)/n$, $0 < \alpha < 1$, $c_k = c(a + \alpha k)$. We assume that the sequence $x_k(a + \alpha k)^{-s}$ is decreasing, the sequence $x_k(a + \alpha k)^{-r}$ is increasing, $x_k = \Lambda(a + \alpha k) \geq 0$. Let x_1 be a given positive number. Define new non-negative variables as follows:

$$\begin{aligned}\alpha_1 &= \frac{x_1}{(a + \alpha)^r}, \\ \alpha_k &= \frac{x_k}{(a + \alpha k)^r} - \frac{x_{k-1}}{(a + \alpha(k-1))^r}, k = 2, ..., n.\end{aligned}$$

Then $x_k/(a+\alpha k)^r = \alpha_1 + ... + \alpha_k$. Since the sequence $x_k(a+\alpha k)^{-s}$ is decreasing, then

$$x_k \leq \left(\frac{a+\alpha k}{a+\alpha(k-1)}\right)^s x_{k-1},\ k = 2, 3, ..., n.$$

This implies

$$\begin{aligned}
\alpha_k &= \frac{x_k}{(a+\alpha k)^r} - \frac{x_{k-1}}{(a+\alpha(k-1))^r} \\
&\leq \left(\frac{(a+\alpha k)^{s-r}}{(a+\alpha(k-1))^s} - \frac{1}{(a+\alpha(k-1))^r}\right) x_{k-1} \\
&= \frac{(a+\alpha k)^{s-r} - (a+\alpha(k-1))^{s-r}}{(a+\alpha(k-1))^s} x_{k-1}.
\end{aligned}$$

Hence

$$\begin{aligned}
\alpha_k &\leq \frac{(a+\alpha k)^{s-r} - (a+\alpha(k-1))^{s-r}}{(a+\alpha(k-2))^s} x_{k-2} \\
&\leq ... \leq \frac{(a+\alpha k)^{s-r} - (a+\alpha(k-1))^{s-r}}{(a+\alpha)^s} x_1.
\end{aligned}$$

We have obtained the upper bound for α_k. Now we have the following optimization problem:

$$z = \max\left(\alpha \sum_{k=1}^{n} c_k \exp\left(-(a+\alpha k)^r(\alpha_1 + ... + \alpha_k)\right)\right)$$

subject to

$$\begin{aligned}
d &= \alpha \sum_{k=1}^{n} \exp\left(-(a+\alpha k)^r(\alpha_1 + ... + \alpha_k)\right), \\
\alpha_1 &= \frac{x_1}{(a+\alpha)^r} > 0, \\
0 &\leq \alpha_k \leq \frac{(a+\alpha k)^{s-r} - (a+\alpha(k-1))^{s-r}}{(a+\alpha)^s} x_1,\ k = 2, ..., n.
\end{aligned}$$

Let c be an increasing function. According to Lemma 1, z achieves its maximum at $\alpha_2 = ... = \alpha_n = 0$. Then $x_k = (a+\alpha k)^r \alpha_1$. Since the sequence

$$\frac{x_k}{(a+\alpha k)^s} = \frac{\alpha_1}{(a+\alpha k)^{s-r}}$$

is decreasing, we have obtained the optimal solution to the above problem. By passing to the limit as $\alpha \to 0$, we find that z achieves its maximum at $\Lambda_r(t) = qt^r$.

Let c be a decreasing function. According to Lemma 1, z achieves its maximum at large values of α_k, *i.e.* at

$$\alpha_k = \frac{(a+\alpha k)^{s-r} - (a+\alpha(k-1))^{s-r}}{(a+\alpha)^s} x_1.$$

This implies

$$\alpha_1 + ... + \alpha_k = \frac{(a+\alpha k)^{s-r}}{(a+\alpha)^s} x_1.$$

Hence

$$x_k = \left(\frac{a+\alpha k}{a+\alpha}\right)^s x_1, \ k = 1, ..., n.$$

Since the sequence

$$\frac{x_k}{(a+\alpha k)^s} = \frac{(a+\alpha k)^{s-r}}{(a+\alpha)^s} x_1$$

is increasing, we have obtained the optimal solution to the above problem. By passing to the limit as $\alpha \to 0$, we obtain that z achieves its maximum at $\Lambda(t) = qt^s$. $\square$

Denote $\Lambda(a,q,t) = (\Gamma(1+q^{-1})\, ta^{-1})^q$.

Corollary 1 *Let c be a monotone function. Then the optimization problem*

$$z = \max \int_0^\infty c(t) \exp(-\Lambda(t)) \mathrm{d}t$$

subject to

$$\Lambda \in \Lambda(r,s), \ \int_0^\infty \exp(-\Lambda(t)) \mathrm{d}t = d,$$

has a solution. If c is decreasing, then z achieves its maximum at $\Lambda_s(t) = \Lambda(d,s,t)$. If c is increasing, then z achieves its maximum at $\Lambda_r(t) = \Lambda(d,r,t)$.

Proof. The proof follows from Lemma 2. If c is decreasing, then $\Lambda_s(t) = qt^s$ and the value of q can be found from the equation $\int_0^\infty \exp(-qt^s)\mathrm{d}t = d$. Hence $q = (\Gamma(1+s^{-1})\, d^{-1})^s$ and $\Lambda_s(t) = \Lambda(d,s,t)$. The second case is proved similarly. $\square$

Lemma 3 *Let c be a monotone function. Then the optimization problem*

$$z = \min \int_a^b c(t) \exp(-\Lambda(t)) \mathrm{d}t$$

subject to $\Lambda \in \Lambda(r,s)$, $\int_a^b \exp(-\Lambda(t))\mathrm{d}t = d$, has a solution. If c is decreasing, then z achieves its minimum at $\Lambda(t) = qt^r$. If c is increasing, then z achieves its minimum at $\Lambda(t) = qt^s$. Here q is a constant.

Proof. Note that $\min Z = \max(-Z)$. Then the proof follows from Lemma 2. $\square$

Corollary 2 *Let c be a monotone function. Then the optimization problem*

$$z = \min \int_0^\infty c(t) \exp(-\Lambda(t)) \mathrm{d}t$$

subject to

$$\Lambda \in \Lambda(r,s), \int_0^\infty \exp(-\Lambda(t))\mathrm{d}t = d,$$

has a solution. If c is decreasing, then z achieves its minimum at $\Lambda_r(t) = \Lambda(d,r,t)$. If c is increasing, then z achieves its minimum at $\Lambda_s(t) = \Lambda(d,s,t)$.

Proof. Similar to the proof for Corollary 1. $\square$

Lemma 4 *Let X be the lifetime and $Y \in \mathcal{H}(r,s)$, where the function Y is defined by $Y(t) = \Pr(X \geq t)$. Then $Z \in \mathcal{H}(r/m, s/m)$, $m \in \mathbf{R}^+$, where the function Z is defined by $Z(t) = \Pr(X^m \geq t)$.*

Proof. Note that $\Pr(X^m \geq t) = \Pr(X \geq t^{1/m}) = \exp(-\Lambda(t^{1/m}))$. Then $\Lambda(t^{1/m})/(t^{1/m})^r = \Lambda(t)/t^r$ is increasing and $\Lambda(t^{1/m})/(t^{1/m})^s = \Lambda(t)/t^s$ is decreasing. $\square$

Lemmas 2 and 3 allow us to analyze the system reliability when lifetimes are bounded.

5 Reliability analysis

Corollaries 1 and 2 play an important role in the imprecise reliability analysis of various systems. They show how to solve problem (3) when we have additional information about lifetime distributions. In the sequel we attempt to formulate the relations between the reliability measures of several systems and their components.

5.1 One component

Theorem 1 *Let $g(t) = t^v$ and $f(t) = t^w$, $v, w \in \mathbf{R}^+$. Suppose that we know the lower $\underline{M}_g$ and upper $\overline{M}_g$ previsions of the gamble g. Moreover, $Y \in \mathcal{H}(r, s)$, where the function Y is defined by $Y(t) = \Pr(g(X) \geq t)$. Denote*

$$\Phi(q) = \Gamma\left(1 + wq^{-1}\right)\left[\Gamma\left(1 + vq^{-1}\right)\right]^{-w/v}.$$

If $v < w$, then lower $\underline{M}_f$ and upper $\overline{M}_f$ previsions of the gamble f are $\underline{M}_f = \Phi(s)\underline{M}_g^{w/v}$, $\overline{M}_f = \Phi(r)\overline{M}_g^{w/v}$. If $v \geq w$, then lower $\underline{M}_f$ and upper $\overline{M}_f$ previsions of the gamble f are $\underline{M}_f = \Phi(r)\underline{M}_g^{w/v}$, $\overline{M}_f = \Phi(s)\overline{M}_g^{w/v}$.

Proof. The natural extension can be written

$$\underline{M}_f = \min_{\Lambda \in \Lambda(r,s)} \int_0^\infty wt^{w-1}\exp(-\Lambda(t))\mathrm{d}t,$$

subject to $a = \int_0^\infty vt^{v-1}\exp(-\Lambda(t))\mathrm{d}t$. Let $x = t^v$. Then we obtain

$$\underline{M}_f = \min_{\Lambda \in \Lambda(r,s)} \frac{w}{v}\int_0^\infty x^{(w/v)-1}\exp(-\Lambda(x^{1/v}))\mathrm{d}x,$$

subject to $a = \int_0^\infty \exp(-\Lambda(x^{1/v}))\mathrm{d}x$. By using Corollary 2 and Lemma 4, we obtain for the case $v < w$

$$\underline{M}_f = \frac{w}{v}\int_0^\infty x^{(w/v)-1}\exp(-\Lambda(a, sv^{-1}, x))\mathrm{d}x.$$

By using Corollary 1 and Lemma 4, we can similarly obtain

$$\overline{M}_f = \frac{w}{v}\int_0^\infty x^{(w/v)-1}\exp(-\Lambda(a, rv^{-1}, x))\mathrm{d}x.$$

Note that $\underline{M}_f$ and $\overline{M}_f$ increase as a increases. By simplifying the above expressions, we complete the proof. The case $v \geq w$ is proved similarly. □

If v and w are integers, then $\underline{M}_g$, $\overline{M}_g$, $\underline{M}_f$, and $\overline{M}_f$ can be regarded as bounds for the v-th and w-th moments of the lifetime.

Example 4 *We know the first moment $a = \underline{M}_g = \overline{M}_g$ of a lifetime which has a IFRA distribution. Then bounds for the w-th moment are determined by $\underline{M}_f = a^w$, $\overline{M}_f = w!a^w$. If $w = 2$, then $\underline{M}_f = a^2$, $\overline{M}_f = 2a^2$.*

Example 5 *We know the v-th moment $a = \underline{M}_g = \overline{M}_g$ of a lifetime which has a IFRA distribution. Then bounds for the first moment are determined by $\underline{M}_f = (v!)^{-1/v}a^{1/v}$, $\overline{M}_f = a^{1/v}$. If $v = 2$, then $\underline{M}_f \simeq 0.707\sqrt{a}$, $\overline{M}_f = \sqrt{a}$.*

5.2 Series systems

A system is called *series* if its lifetime is given by $\min_{i=1,\ldots,n} x_i$.

Theorem 2 *A series system consists of n independent components with lower and upper MTTFs $\underline{a}_i$ and $\overline{a}_i$, $0 \leq \underline{a}_i \leq \overline{a}_i$, $i = 1, \ldots, n$. Suppose that the i-th component lifetime distribution belongs to $\mathcal{H}(r_i, s_i)$, $i = 1, \ldots, n$. Then the lower $\underline{M}$ and upper $\overline{M}$ system MTTFs are*

$$\underline{M} = \int_0^\infty \prod_{i=1}^n \exp\left(-\Lambda(\underline{a}_i, r_i, t)\right)\mathrm{d}t,\ \overline{M} = \int_0^\infty \prod_{i=1}^n \exp\left(-\Lambda(\overline{a}_i, s_i, t)\right)\mathrm{d}t.$$

Proof. Let $\underline{a}_i \leq a_i \leq \overline{a}_i$. The MTTF of the series system is computed as follows:

$$\begin{aligned} M(a_1, \ldots, a_n) &= \int_0^\infty \prod_{i=1}^n \exp(-\Lambda_i(t))\mathrm{d}t \\ &= \int_0^\infty \exp(-\Lambda_j(t)) \prod_{i=1, i\neq j}^n \exp(-\Lambda_i(t))\mathrm{d}t. \end{aligned}$$

Denote $c(t) = \prod_{i=1,i\neq j}^{n} \exp(-\Lambda_i(t))$. The function c is decreasing in t for all $j = 1, ..., n$. Moreover, the function $M(a_1, ..., a_n)$ increases as a_i increases, $i = 1, ..., n$. By using Corollaries 1 and 2, we complete the proof. □

Corollary 3 *If $\mathcal{H}(r_i, s_i) = \mathcal{H}(r, s)$ for all $i = 1, ..., n$, then*

$$\underline{M} = \left(\sum_{i=1}^{n} \frac{1}{\underline{a}_i^r}\right)^{-1/r}, \ \overline{M} = \left(\sum_{i=1}^{n} \frac{1}{\overline{a}_i^s}\right)^{-1/s}.$$

It follows from Corollary 3 that if $r = 1$, $s = +\infty$ (IFRA distributions), then

$$\underline{M} = \left(\sum_{i=1}^{n} \frac{1}{\underline{a}_i}\right)^{-1}, \ \overline{M} = \min_{i=1,...,n} \overline{a}_i.$$

If $r = 0$, $s = 1$ (DFRA distributions), then

$$\underline{M} = 0, \ \overline{M} = \left(\sum_{i=1}^{n} \frac{1}{\overline{a}_i}\right)^{-1}.$$

Let us return to the example of judgements that was presented in the Introduction of the paper. The initial judgements allow us to conclude that the lower and upper MTTFs of the two-component series system are 0 and 5, respectively. After obtaining additional information, we assume that $r_A \simeq 0.9$ (long infancy period), $s_A = 2$ (see Example 3), i.e. $\Lambda_A \in \Lambda(0.9, 2)$. Similarly, $\Lambda_B \in \Lambda(1, +\infty)$ (only the wear-out period). It follows from Theorem 2 that the lower and upper MTTFs of the system are 2.68 and 4.69, respectively. Note that if we take $r_A \simeq 0.5$ (middle infancy period), then the MTTFs are 1.6 and 4.69.

5.3 Parallel systems

A system is called *parallel* if its lifetime is given by $\max_{i=1,...,n} x_i$.

Theorem 3 *A parallel system consists of n independent components with the lower and upper MTTFs $\underline{a}_i$ and $\overline{a}_i$, $0 \leq \underline{a}_i \leq \overline{a}_i$, $i = 1, ..., n$. Suppose that the i-th component lifetime* *distribution belongs to $\mathcal{H}(r_i, s_i)$, $i = 1, ..., n$. Then the lower $\underline{M}$ and upper $\overline{M}$ system MTTFs are*

$$\begin{aligned}\underline{M} &= \int_0^\infty \left(1 - \prod_{i=1}^n \left(1 - \exp\left(-\Lambda(\underline{a}_i, s_i, t)\right)\right)\right) \mathrm{d}t, \\ \overline{M} &= \int_0^\infty \left(1 - \prod_{i=1}^n \left(1 - \exp\left(-\Lambda(\overline{a}_i, r_i, t)\right)\right)\right) \mathrm{d}t.\end{aligned}$$

Proof. Let $\underline{a}_i \leq a_i \leq \overline{a}_i$. The MTTF of the parallel system is

$$\begin{aligned}M(a_1, ..., a_n) &= \int_0^\infty \left(1 - \prod_{i=1, i\neq j}^n (1 - \exp(-\Lambda_i(t)))\right) \mathrm{d}t \\ &\quad + \int_0^\infty \exp(-\Lambda_j(t)) \prod_{i=1, i\neq j}^n (1 - \exp(-\Lambda_i(t)))\mathrm{d}t.\end{aligned}$$

Denote $c(t) = \prod_{i=1,i\neq j}^n (1 - \exp(-\Lambda_i(t)))$. The function c is increasing in t for all $j = 1, ..., n$. Moreover, the function $M(a_1, ..., a_n)$ is increasing in a_i. By using Corollaries 1 and 2, we complete the proof. □

Corollary 4 *If $\mathcal{H}(r_i, s_i) = \mathcal{H}(r, s)$ for all $i = 1, ..., n$, then*

$$\begin{aligned}\underline{M} &= \sum_{i=1}^n \underline{a}_i - \sum_{i<j} \left(\frac{1}{\underline{a}_i^s} + \frac{1}{\underline{a}_j^s}\right)^{-1/s} + ... + (-1)^{n-1} \left(\sum_{i=1}^n \frac{1}{\underline{a}_i^s}\right)^{-1/s}, \\ \overline{M} &= \sum_{i=1}^n \overline{a}_i - \sum_{i<j} \left(\frac{1}{\overline{a}_i^r} + \frac{1}{\overline{a}_j^r}\right)^{-1/r} + ... + (-1)^{n-1} \left(\sum_{i=1}^n \frac{1}{\overline{a}_i^r}\right)^{-1/r}.\end{aligned}$$

It follows from Corollary 4 that if $r = 1$, $s = +\infty$ (IFRA distributions), then

$$\begin{aligned}\underline{M} &= \max_{i=1,...,n} \underline{a}_i, \\ \overline{M} &= \sum_{i=1}^n \overline{a}_i - \sum_{i<j} \left(\frac{1}{\overline{a}_i} + \frac{1}{\overline{a}_j}\right)^{-1} + ... + (-1)^{n-1} \left(\sum_{i=1}^n \frac{1}{\overline{a}_i}\right)^{-1}.\end{aligned}$$

If $r = 0$, $s = 1$ (DFRA distributions), then

$$\begin{aligned}\underline{M} &= \sum_{i=1}^n \underline{a}_i - \sum_{i<j} \left(\frac{1}{\underline{a}_i} + \frac{1}{\underline{a}_j}\right)^{-1} + ... + (-1)^{n-1} \left(\sum_{i=1}^n \frac{1}{\underline{a}_i}\right)^{-1}, \\ \overline{M} &= \sum_{i=1}^n \overline{a}_i.\end{aligned}$$

5.4 Monotone systems

A system is called *monotone* if it does not become better by a failure of a component. It should be noted that generally for arbitrary monotone systems, the function c can be non-monotone and we can not directly use the results of Section 4. In that case, the minimal paths and cut sets presentation or modular decomposition technique can be employed to calculate the system reliabilities. A *minimal path* of a system is a minimal set of components such that if these components work, the system works. A *minimal cut* is a minimal set of components such that if these components fail, the system fails. Suppose that a monotone system has p minimal paths $P_1, ..., P_p$ containing $m_1, ..., m_p$ components, respectively, and k minimal cut sets $K_1, ..., K_k$. According to Barlow and Proschan (1975), the system lifetime $g(\mathbf{X})$ is given by

$$g(\mathbf{X}) = \max_{1 \leq j \leq p} \min_{i \in P_j} x_i = \min_{1 \leq j \leq k} \max_{i \in K_j} x_i.$$

Suppose a monotone system consists of n independent components with the lower and upper MTTFs $\underline{a}_i$ and $\overline{a}_i$, $0 \leq \underline{a}_i \leq \overline{a}_i$, $i = 1, ..., n$, and the i-th component lifetime distribution belongs to $\mathcal{H}(r_i, s_i)$. Then the following inequalities for the lower $\underline{M}$ and upper $\overline{M}$ system MTTFs hold:

$$\begin{aligned}
\underline{M} &\geq \max_{1 \leq j \leq p} \int_0^{\infty} \prod_{i \in P_j} \exp\left(-\Lambda(\underline{a}_i, r_i, t)\right) \mathrm{d}t, \\
\overline{M} &\leq \min_{1 \leq j \leq k} \int_0^{\infty} \left(1 - \prod_{i \in K_j} \left(1 - \exp\left(-\Lambda(\overline{a}_i, r_i, t)\right)\right)\right) \mathrm{d}t.
\end{aligned}$$

Another method for calculating system reliabilities is the modular decomposition technique. The method is based on subdivision of a system into series and parallel modules. By computing the lower and upper MTTFs of modules (see Theorems 2 and 3), and by determining the parameters r and s of the obtained lifetime distribution classes for each module (see Theorems 4 and 5), we can consider each module as one component for which the MTTFs and parameters r, s are known. Let us find the values of r and s for the series and parallel systems.

Theorem 4 *Let a series system consist of n independent components. Suppose that the i-th component lifetime distribution belongs to $\mathcal{H}(r_i, s_i)$, $i = 1, ..., n$. Then the system lifetime distribution belongs to $\mathcal{H}(r, s)$, where $r = \min_{1\le i\le n} r_i$, $s = \max_{1\le i\le n} s_i$.*

Proof. For a series system $\Lambda(t) = \sum_{i=1}^{n} \Lambda_i(t)$. Then the proof is obvious from the following:

$$\frac{\Lambda(t)}{t^r} = \sum_{i=1}^{n} \frac{\Lambda_i(t)}{t^{r_i}} t^{r_i - r}, \ \frac{\Lambda(t)}{t^s} = \sum_{i=1}^{n} \frac{\Lambda_i(t)}{t^{s_i}} \frac{1}{t^{s_i - s}}.$$

□

Theorem 5 *Let a parallel system consist of n independent components. Suppose that the i-th component lifetime distribution belongs to $\mathcal{H}(r_i, s_i)$, $i = 1, ..., n$. Then the system lifetime distribution belongs to $\mathcal{H}(r, s)$, where $r = \min_{1\le i\le n} r_i$, $s = \sum_{i=1}^{n} s_i$.*

Proof. For a parallel system $H(t) = 1 - \prod_{i=1}^{n}(1 - H_i(t))$ and $\Lambda(t) = -\ln H(t)$. Hence

$$\lambda(t) = \frac{\sum_{j=1}^{n} \prod_{i=1, i\neq j}^{n} (1 - H_i(t)) H_j(t) \lambda_j(t)}{1 - \prod_{i=1}^{n}(1 - H_i(t))}.$$

Introduce the function $\varphi(t) = t\lambda(t)/\Lambda(t)$. Then

$$\varphi(t) = \frac{-t \sum_{j=1}^{n} \prod_{i=1, i\neq j}^{n} (1 - H_i(t)) H_j(t) \lambda_j(t)}{H(t) \ln H(t)}.$$

Let us consider the maximum and minimum of φ over all distributions $H_i \in \mathcal{H}(r_i, s_i)$. From Property 4, we can write $r_j \Lambda_j(t) \le t\lambda_j(t) \le s_j \Lambda_j(t)$. This implies that $\underline{\varphi}(t) \le \varphi(t) \le \overline{\varphi}(t)$, where

$$\begin{aligned}
\underline{\varphi}(t) &= \frac{-\sum_{j=1}^{n} \prod_{i=1, i\neq j}^{n} (1 - H_i(t)) H_j(t) r_j \Lambda_j(t)}{H(t) \ln H(t)}, \\
\overline{\varphi}(t) &= \frac{-\sum_{j=1}^{n} \prod_{i=1, i\neq j}^{n} (1 - H_i(t)) H_j(t) s_j \Lambda_j(t)}{H(t) \ln H(t)}.
\end{aligned}$$

If we denote $x_i = 1 - H_i(t)$, then $\Lambda_i(t) = -\ln(1 - x_i)$. By dividing the numerator and denominator by $\prod_{i=1}^{n} x_i$, we obtain

$$\underline{\varphi}(t) = \sum_{j=1}^{n} r_j Y_j / Y, \ \overline{\varphi}(t) = \sum_{j=1}^{n} s_j Y_j / Y,$$

where

$$Y_j = \frac{(1-x_j)\ln(1-x_j)}{x_j},$$
$$Y = \frac{(1-x_1\cdots x_n)\ln(1-x_1\cdots x_n)}{x_1\cdots x_n}.$$

Let us consider the function $\psi(t) = \sum_{j=1}^{n} k_j Y_j / Y$, where k_j is a positive constant. It can be easily proved that the function ψ decreases as x_j increases, $0 < x_j < 1$, $j = 1,...,n$. Consequently, ψ achieves its maximum as $x_j \to 0$. Since $\lim_{x\to 0}(1-x)x^{-1}\ln(1-x) = -1$, then $\max \overline{\varphi}(t) = \sum_{j=1}^{n} s_j$. The limit value of ψ depends on the order of numbers j for which $x_j \to 1$. If j_0 is the last number, then the limit value of ψ is r_{j_0}. This implies that $\min \underline{\varphi}(t) = \min_{1\le i\le n} r_i$. This completes the proof. □

5.5 Bounded lifetimes

By using Lemmas 2 and 3, we can obtain lower and upper MTTFs of systems with bounded component lifetimes. Let $\underline{M}(x_i) = \underline{a}_i$ and $\overline{M}(x_i) = \overline{a}_i$ be the lower and upper MTTFs of the i-th component, $i = 1,...,n$. Suppose that $0 \le x_i \le T_i$, $i = 1,...,n$. In this case, explicit expressions for the system reliability can be obtained only for special cases. For example, the lower and upper MTTFs of a series system consisting of n independent components with unknown lifetime distributions ($\mathcal{H}(0,\infty)$) are

$$\underline{M} = \left(\min_{i=1,...,n} T_i\right)\left(\prod_{i=1}^{n}\frac{\underline{a}_i}{T_i}\right),\ \overline{M} = \min_{i=1,...,n} \overline{a}_i.$$

Denote $T = \max_{i=1,...,n} T_i$. The lower and upper MTTFs of a parallel system consisting of n independent components with unknown lifetime distributions ($\mathcal{H}(0,\infty)$) are

$$\underline{M} = \max_{i=1,...,n} \underline{a}_i,\ \overline{M} = T\left(1-\prod_{i=1}^{n}\left(1-\frac{\overline{a}_i}{T_i}\right)\right).$$

The proof of the above bounds can be found in (Utkin and Gurov, 1999). Let us give some important properties of series and parallel systems consisting of independent components with bounded lifetimes.

1. If the component lifetime distributions are unknown and $T_i \to \infty$ for all $i = 1, ..., n$, then the condition of independence of the system components (in the sense of the definition given in Section 2) does not influence the lower and upper MTTFs of systems. For example, the MTTFs of a series system are $\underline{M} = 0$, $\overline{M} = \min_{i=1,...,n} \overline{a}_i$. These expressions coincide with expressions obtained from the natural extension without the independence assumption (Utkin, 1998).

2. If the component lifetime distributions are unknown and $n \to \infty$, then the condition of independence of the system components (in the sense of the definition given in Section 2) does not influence the lower and upper MTTFs of systems. For example, the limiting MTTFs of a series system as $n \to \infty$ are $\underline{M} = 0$, $\overline{M} = \min_{i=1,...,n} \overline{a}_i$. These expressions coincide with expressions obtained from the natural extension without the independence assumption (Utkin, 1998).

6 Discrete lifetime distributions

In this section we show that the results obtained in previous sections for continuous lifetime distributions can be applied to discrete lifetime distributions. Discrete lifetimes usually arise through grouping or finite-precision measurement of continuous time phenomena. They may also be found naturally in those cases where failure may occur only at instants of shock. For a random lifetime X taking positive integral values, let $G(k) = \Pr(X \geq k)$. Let $R(k) = -\ln G(k)$. Let us define a distribution class $\mathcal{G}(r, s)$ as follows. A probability distribution belongs to $\mathcal{G}(r, s)$ if $R(k)/k^r$ increases and $R(k)/k^s$ decreases as k increases. In that case we will write $R \in R(r, s) = \{R : \exp(-R) \in \mathcal{G}(r, s)\}$.

Let x_q be the solution of the equation $\sum_{k \geq 0} \exp(-k^q x_q) = a$.

Lemma 5 *Let c be a monotone function, $k = 0, 1, \ldots$. Then the optimization problem:*

$$z = \max\left(\sum_{k\geq 0} c(k)\exp(-R(k))\right)$$

subject to

$$R \in R(r,s),\ \sum_{k\geq 0} \exp(-R(k)) = a,$$

has a solution. If c is decreasing, then z achieves its maximum at $R_s(k) = k^s x_s$. If c is increasing, then z achieves its maximum at $R_r(k) = k^r x_r$.

Proof. Similar to the proof of Lemma 2. □

Lemma 6 *Let $c(k)$ be a monotone function, $k = 0, 1, \ldots$. Then the optimization problem:*

$$z = \min\left(\sum_{k\geq 0} c(k)\exp(-R(k))\right)$$

subject to

$$R \in R(r,s),\ \sum_{k\geq 0} \exp(-R(k)) = a,$$

has a solution. If c is decreasing, then z achieves its maximum at $R_r(k) = k^r x_r$. If c is increasing, then z achieves its maximum at $R_s(k) = k^s x_s$.

Proof. Similar to the proof of Lemma 3. □

The reliability assessments for discrete lifetime distributions are similar to assessments for continuous lifetime distributions. For example, the MTTFs of a series system consisting of n independent components whose component lifetime distributions belong to $\mathcal{G}(r_i, s_i)$, $i = 1, \ldots, n$, are

$$\underline{M} = \sum_{k\geq 0}\prod_{i=1}^{n} \exp\left(-k^{r_i} x_{r_i}\right),\ \overline{M} = \sum_{k\geq 0}\prod_{i=1}^{n} \exp\left(-k^{s_i} x_{s_i}\right),$$

where x_{r_i} is the solution of the equation $\sum_{k\geq 0} \exp(-k^{r_i} x_{r_i}) = \underline{a}_i$, and x_{s_i} is the solution of the equation $\sum_{k\geq 0} \exp(-k^{s_i} x_{s_i}) = \overline{a}_i$.

7 Conclusion

In this paper, we have shown how additional information about the ageing aspect of lifetime distributions and the independence of system components can be used for analyzing system reliability. The results generalize the reliability models based on natural extension (Utkin, 1998; Utkin and Gurov, 1998; Utkin and Gurov, 1999). It should be noted that the results in the paper also generalize the reliability bounds presented by Barlow and Proschan (1975) for distributions with the increasing and decreasing failure rate distributions.

We have found the solution of the optimization problems in an explicit form for several special cases. This simplifies usage of the present approach in practice and makes it understandable from an engineering point of view. The results allow us to compute the reliability of various unrepairable systems whose reliability characteristics are represented by the lower and upper MTTFs. However, it should be noted that the class of analyzed systems can be easily extended, for instance, to the case when we know only some moments of the lifetime.

The proposed lifetime distribution classes can cover a wide variety of kinds of partial and precise reliability information. At the same time, further work is needed to develop efficient statistical methods for calculating parameters of the distribution classes, which may lead to new questions and ideas. One approach is to determine bounds for the probability distributions and densities of a class and to maximize an "average" likelihood function with parameters r and s. Our computational results show that the values of r and s obtained using this approach agree with some theoretical properties of classes.

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