CAUTIOUS RELIABILITY ANALYSIS OF MULTI-STATE AND CONTINUUM-STATE SYSTEMS BASED ON THE IMPRECISE DIRICHLET MODEL

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Abstract

Cautious reliability estimates of multi-state and continuum-state systems are studied in the paper under condition that initial data about reliability of components are given in the form of interval-valued observations, measurements or expert judgments. The interval-valued information is processed by means of a set of the imprecise Dirichlet model which can be regarded as a set of Dirichlet distributions. The developed model of reliability provides cautious reliability measures when the number of observations or measurements is rather small. It can be viewed as an extension of models based on random set theory and robust statistical models. A numerical example illustrates the proposed model and an algorithm for computing the system reliability.

Keywords: Multi-state system; continuum-state system; Dirichlet distribution; random set theory; robust model; Bayesian updating; series and parallel systems.

1 Introduction

Much attention have been focused on multi-state (MS) and continuum-state (CS) systems due to their wide applicability. These systems can be regarded as a generalization of binary systems. A lot of methods[1, 2, 3, 4] for reliability analysis of various MS and CS systems have been recently proposed in the literature and a comprehensive review of many methods can be found in a book written by Levitin and Lisnianski[5]. However, most approaches to reliability analysis assume that precise probability distributions (probability density functions) on states of MS (CS) components are known. But it is difficult to expect that such the complete information is always available. Therefore, the approaches and models taking into account possible incompleteness of information about reliability behavior of MS and CS systems are very important. Some models of the MS system reliability using the *imprecise probability theory* (also called the theory of lower previsions[6], the theory of interval statistical models[7], the theory of interval probabilities[8]) have been proposed by Utkin and Gurov[9], Utkin and Kozine[10]. These models assume that the initial information about states of components is partial and represented as a set of probabilistic constraints which produce a set of possible probability distributions of the component states. These constraints usually restrict possible values of expectations of some functions of the component states. It should be noted that various kinds of reliability judgments[11, 12] can be represented by constraints for expectations (by lower and upper expectations), in particular, probabilities (expectations of indicator functions), mean levels of system or component performance, comparative judgments, etc. As a result, the obtained system reliability measures become interval-valued. At the same time, these models require having probabilistic expert judgments at hand and this is a very strong restriction due to a lack of such data in many cases. Another way for taking into account the fact that there are no precise probabilities of states was proposed by Lindqvist and Langseth[13]. They assumed that

the probabilities of states are random variables themselves defined on the unit simplex and governed by the Dirichlet distribution. However, the authors assumed that parameters of the Dirichlet distributions are precisely known that often may be not a case.

The present paper studies a model dealing with interval-valued observations or measurements of the component states. It is no secret that the precise observations of states can not be obtained for many real systems, especially, for CS systems, due to some imperfection of diagnostic and measuring equipment and methods. Therefore, the corresponding approaches processing such initial data have to be developed. One of the possible frameworks for taking into account interval-valued observations or measurements is *Dempster-Shafer* or *random set theory*[14, 15]. It is worth noting that this theory has been applied to reliability analysis of some systems [16, 17]. However, the methods proposed and Dempster-Shafer theory itself do not take into account the fact that the number of observations or measurements might be very small and reliability conclusions using this theory might be too risky in this case. Therefore, the main objective of the present paper is to develop an approach taking into account the above difficulties of the available models and providing the cautious reliability analysis of MS and CS systems by interval-valued observations or measurements of the component states. The main tool for achieving this objective is the so-called imprecise Dirichlet model [18] which can be regarded as a set of Dirichlet distributions. From this point of view, the developed model can be viewed as an extension of some results of the work[13] to unreliable interval-valued measurements.

The paper is organized as follows. A short description of a general model of MS and CS systems is given in Section 2. Section 3 studies a method for uncertainty modelling of the component reliability on the basis of the imprecise Dirichlet model. Several interpretations and a updating procedure of the obtained lower and upper probabilities are also given in this section. General expressions for reliability measures of MS systems are proposed in Section 4. These expressions are simplified for series and parallel MS systems in the same section. Reliability analysis of CS systems is provided in Section 5. The theoretical results are illustrated by a numerical example in Section 6. Propositions stated in the paper are proved in Appendix.

2 General model of multi-state and continuum-state systems

Let $L_i = \{S_1^{(i)}, ..., S_m^{(i)}\}$ be the set representing levels of the i-th component performance ranging from perfect functioning $\sup L_i$ to complete failure $\inf L_i$. A *general* model of the structure function of a system consisting of n MS components has been proposed by Montero et al.[4]. It can be written as $S : L_1 \times ... \times L_n \to L_0$. In order to give the reader the essence of the subject analyzed and make all the formulas more readable, we will assume $L_0 = L_1 = ... = L_n = L$. If $L = \{0, 1\}$, $i = 0, ..., n$, we have a classical *binary* system; if $L = \{0, 1, \dots, m\}$, we have a *MS* system; if $L = [0, T]$, $T \in \mathbf{R}^+$, we have a *CS* system. At arbitrary time t the i-th component may be in a state $x_i(t)$. This implies that the component is described by the random process $\{x_i(t), t \geq 0\}$, $x_i(t) \in L$. Then the probability distribution function of the i-th component states at time t is defined as the mapping $F_i : L \to [0, 1]$ such that $F_i(r) = \Pr\{x_i(t) \geq r\}$ $\forall r \in L$. The probability that the i-th component is in the r-th state for MS systems is defined as $\pi_i(r) = \Pr\{x_i(t) = r\}$. For CS system, we also define the density function of states as $f_i(r) = \mathrm{d}F_i(r)/\mathrm{d}r$. The state of the system is determined by states of its n components, i.e.,

$$S = S(X) = S(x_1, \dots, x_n) \in L.$$

Then the probability distribution function of the system states at time t is defined as the mapping $F : L \to [0, 1]$ such that $F(r) = \Pr\{S(X) \geq r\}$ $\forall r \in L$. In the following, we shall omit variable t for brevity.

A system of n components with the structure function S is called a *monotone* MS system if $S(X)$ is increasing in each argument and $S(r, \dots, r) = r$ $\forall r \in L$. The basic properties of monotone MS systems were studied by Barlow and Wu[2].

The reliability measures of the MS system also can be represented in the form of expectations. For example, the probability $\Pr\{S(X) \geq r\}$ is written as

$$\Pr\{S(X) \geq r\} = \mathbb{E}\mathbf{1}_{[r,m]}(S(X)).$$

Here $\mathbf{1}_{[a,b]}(x)$ is the indicator function taking the value 1 if $x \in [a, b]$ and 0 if $x \notin [a, b]$. The *mean level of system performance* h is determined as $h = \mathbb{E}S(X)$.

3 Uncertainty model of multi-state components

The standard approach to reliability analysis of MS systems is to find the probabilities $\pi_i(r)$ of states of every component and to calculate the system reliability measures by using the system function $S(X)$. The usual procedure in the common case of unknown probabilities $\pi_i(r)$ is to use the empirical probabilities $\pi_i(r) = n_i(r)/N_i$ as if they were precisely known chances. Here $n_i(r)$ is the number of observations of the r-th state in N_i trials. This is not always suitable: (i) the point estimate $\pi_i(r)$ might be not available; (ii) the possible small number of statistical data makes the statistical inference too incautious. Therefore, we propose an uncertainty model of a component under conditions of lack of sufficient statistical data and of possible lack of point estimates.

Suppose that there is a set of K_i bounds for the i-th component states in the form of intervals (subsets of states) $A_1^{(i)}$, $A_2^{(i)}$, ..., $A_{K_i}^{(i)}$ such that $A_j^{(i)} \subseteq L$. All intervals are provided by experts or resulted from observations and measurements. At that, it is assumed that a "true" state might be inside the interval $A_j^{(i)}$.

3.1 Imprecise Dirichlet model

Let $L = \{0, ..., m\}$ be a set of states of the i-th component regarded as the possible outcomes. Assume the *standard multinomial model*: N point-valued or "crisp" observations are independently chosen from L with an identical probability distribution π_{ij} for $j = 0, ..., m$, where each $\pi_{ij} \geq 0$ and $\sum_{j=0}^{m} \pi_{ij} = 1$. Denote $\pi^{(i)} = (\pi_{i0}, ..., \pi_{im})$. Let n_{ij} denote the number of observations of state j of the i-th component in the N trials, so that $n_{ij} \geq 0$ and $\sum_{j=0}^{m} n_{ij} = N$. Under the above assumptions the random variables $n_{i0}, ..., n_{im}$ have a multinomial distribution and the observed multinomial likelihood function generated by the data $\mathbf{n}^{(i)} = (n_{i0}, ..., n_{im})$ is

$$L(\mathbf{n}^{(i)}|\pi^{(i)}) \propto \prod_{j=0}^{m} (\pi_{ij})^{n_{ij}} .$$

Suppose that the probabilities $\pi^{(i)}$ of states of the i-th component are random variables governed by the *Dirichlet* (s, α) *distribution* with parameters $\alpha = (\alpha_1, ..., \alpha_m)$, which has probability density function[19]

$$p_i(\pi^{(i)}) = \Gamma(s) \left(\prod_{j=0}^{m} \Gamma(s\alpha_j) \right)^{-1} \prod_{j=0}^{m} \pi_{ij}^{s\alpha_j - 1}.$$

Here the parameter $\alpha_j \in (0, 1)$ is the mean of π_{ij} under the Dirichlet prior; the hyperparameter $s > 0$ determines the influence of the prior distribution on posterior probabilities; the vector $\pi^{(i)}$ belongs to the interior of the K-dimensional unit simplex denoted by $S(1, K)$; $\Gamma(\cdot)$ is the Gamma-function which satisfies $\Gamma(x + 1) = x\Gamma(x)$ and $\Gamma(1) = 1$.

When multiplied by the multinomial likelihood function $L(\mathbf{n}^{(i)}|\pi^{(i)})$, the Dirichlet (s, α) prior density generates a posterior density function

$$p(\pi^{(i)}|\mathbf{n}^{(i)}) \propto p(\pi^{(i)}) L(\mathbf{n}^{(i)}|\pi^{(i)}) = \prod_{j=0}^{m} \pi_{ij}^{n_{ij} + s\alpha_j - 1},$$

which is seen to be the probability density function of a Dirichlet $(N + s, \alpha^*)$ distribution, where $\alpha_j^* = (n_{ij} + s\alpha_j)/(N + s)$.

A property of the Dirichlet distribution is that its marginal distributions are also Dirichlet distributions. In particular, its univariate marginals are Beta distributions ($m = 2$) and the Dirichlet distribution can be seen as a multivariate generalization of the Beta distribution.

Walley[18] pointed out several reasons for using a set of Dirichlet distributions to model prior ignorance about probabilities $\pi^{(i)}$:

1. Dirichlet prior distributions are mathematically tractable because they generate Dirichlet posterior distributions;

2. sets of Dirichlet distributions are very rich, because they produce the same inferences as their convex hulls and any prior distribution can be approximated by a finite mixture of Dirichlet distributions;

3. the most common Bayesian models for prior ignorance about probabilities $\pi^{(i)}$ are Dirichlet distributions.

The *imprecise Dirichlet model* (IDM) is defined by Walley [18] as the set of all Dirichlet (s, α) distributions such that $\alpha \in S(1, m)$. For the IDM , the *hyperparameter* s determines how quickly upper and lower probabilities of events converge as statistical data accumulate. Walley[18] defined s as a number of observations needed to reduce the imprecision (difference between upper and lower probabilities) to half its initial value. Smaller values of s produce faster convergence and stronger conclusions, whereas large values of s produce more cautious inferences. At the same time, the value of s must not depend on m or a number of observations. The detailed discussion concerning the parameter s and the IDM can be found in several papers [20, 21, 18].

Let A be any non-trivial subset of a sample space $L = \{S_1, ..., S_m\}$, and let $n(A)$ denote the observed number of occurrences of A in the N trials, $n(A) = \sum_{S_j \in A} n_j$. Then, according to[18], the predictive probability $P(A|s)$ under the Dirichlet posterior distribution is

$$P(A|s) = \frac{n(A) + s\alpha(A)}{N + s},$$

where $\alpha(A) = \sum_{S_j \in A} \alpha_j$.

By maximizing and minimizing α_j under restriction $\alpha \in S(1, m)$, we obtain the posterior lower and upper probabilities of A:

$$\underline{P}(A|s) = \frac{n(A)}{N + s},\ \overline{P}(A|s) = \frac{n(A) + s}{N + s}.$$

If $A = L$, then $\underline{P}(A|s) = \overline{P}(A|s) = 1$. Note that the IDM is reduced to the "precise" Dirichlet distribution by taking $s = 0$.

Why and when should we use the IDM instead of the "precise" Dirichlet distribution? Suppose that we toss a coin five times and have 3 heads and 2 tails. By using the "precise" Dirichlet distribution (the Dirichlet distribution is none other than the beta-distribution in case of two possibilities $m = 2$), we get $P(heads) = 3/5 \neq 1/2$. At the same time, if we take the IDM with $s = 1$, then $\underline{P}(heads|1) = 3/6$ and $\overline{P}(heads|1) = 4/6$. It can be seen from these results that

$$\underline{P}(heads|1) \leq 1/2 \leq \overline{P}(heads|1).$$

In other words, the IDM provides lower and upper bounds for probabilities of events when the number of observations is rather small and the resulting inference might be too incautious.

3.2 Sets of models produced by unreliable observations

It is worth noticing that the imprecise Dirichlet distribution as well as the Dirichlet distribution works with "crisp" observations when we are able to observe separate states. By observing the subset $A_j^{(i)}$ of the component states and assuming that one of the states in this interval is "true" [1], we can not exactly indicate this "true" state and every state belonging to $A_j^{(i)}$ may equally be "true" except the case when $A_j^{(i)}$ consists of one element. This implies that it is impossible to construct a unique multinomial model of observations and we should consider some set of multinomial models and the corresponding Dirichlet distributions. Moreover, every new subset $A_j^{(i)}$ produces some additional set of the Dirichlet distributions. Denote the k-th possible vector of "crisp" states by $\mathbf{n}_k = (n_{k0}, ..., n_{km})$ and the total number of such the vectors Q. We omit the index i when it is not important. Here $\sum_{j=0}^{m} n_{kj} = K_i$. For example, suppose that we have a 4-state component and two observations $A_1^{(i)} = \{1, 2\}$, $A_2^{(i)} = \{0, 1\}$ of states at time t. Then the possible observations of "crisp" states are shown in Table 1.

[1] A component can not simultaneously be in two or more states. It can be only in one state. By observing a subset of the component, we can say that it has to be in some state which is unknown, but we call this state "true".

Table 1: Possible "crisp" observations

states	0	1	2	3
$\mathbf{n}_1$	0	1	1	0
$\mathbf{n}_2$	1	1	0	0
$\mathbf{n}_3$	1	0	1	0
$\mathbf{n}_4$	0	2	0	0

The above can be represented by means of the well-known urn model. Suppose that there are $m+1$ urns $u_0, ..., u_m$ containing balls with numbers $0, ..., m$, respectively. We randomly choose a subset $A_j^{(i)}$ of l_j urns such that $A_j^{(i)} = \{u_k : k \in J_{ij}\}$, where J_{ij} is a set of indices. Then we take randomly one ball from the urns numbered by elements of J_{ij}. The same procedure is repeated K_i times, i.e., for each of the observed subsets. What can we say about possible numbers of balls chosen from each urn now? It is obvious that there exist different combinations of numbers of balls except the case when $l_j = 1$ for $j = 1, ..., K_i$, i.e., all sets $A_j^{(i)}$ consist of one element. Suppose that the number of the possible combinations is M. If to assume that the subsets $A_j^{(i)}$ are independently chosen from the set of all subsets of urns and the probability of selecting a ball from the i-th urn is π_i, then every combination of balls produces the standard multinomial model. Moreover, we can not prefer one model over another.

For every model (every vector $\mathbf{n}_k$), the probability of an arbitrary subset $A \subseteq L$ depends on $\mathbf{n}_k$, that is, we can find $P(A|\mathbf{n}_k)$. So far as all models are equivalent, even by precise probabilities of all states only lower and upper probabilities of A can be computed

$$\underline{P}(A) = \min_{k=1,...,Q} P(A|\mathbf{n}_k), \quad \overline{P}(A) = \max_{k=1,...,Q} P(A|\mathbf{n}_k).$$

In particular, if all subsets $A_j^{(i)}$ consist of single components, then $Q = 1$ and

$$\underline{P}(A) = P(A|\mathbf{n}_k), \quad \overline{P}(A) = P(A|\mathbf{n}_k).$$

3.3 Computing the probabilities $\underline{P}(A|s)$ and $\overline{P}(A|s)$

By using the IDM , we can write the lower $\underline{P}(A|s)$ and upper $\overline{P}(A|s)$ probabilities that the "true" state of the i-th component belong to subset A as follows:

$$\underline{P}(A|s) = \min_{k=1,...,Q} \inf_{\alpha \in S(1,m+1)} \frac{n_k(A) + s\alpha(A)}{K_i + s},$$

$$\overline{P}(A|s) = \max_{k=1,...,Q} \sup_{\alpha \in S(1,m+1)} \frac{n_k(A) + s\alpha(A)}{K_i + s},$$

where

$$\alpha(A) = \sum_{j \in A} \alpha_j, \quad n_k(A) = \sum_{j \in A} n_{kj}.$$

Now we have to find $n_k(A)$ and $\alpha(A)$. The lower and upper probabilities $\underline{P}(A|s)$ and $\overline{P}(A|s)$ can be rewritten as follows:

$$\underline{P}(A|s) = \frac{\min\limits_{k=1,...,Q} n_k(A) + s \cdot \inf\limits_{\alpha \in S(1,m+1)} \alpha(A)}{K_i + s},$$

$$\overline{P}(A|s) = \frac{\max\limits_{k=1,...,Q} n_k(A) + s \cdot \sup\limits_{\alpha \in S(1,m+1)} \alpha(A)}{K_i + s}.$$

Proposition 1 *Denote*

$$Z_1(A) = \min_{k=1,\dots,Q} n_k(A) = \sum_{j:A_j^{(i)} \subseteq A} 1,$$

$$Z_2(A) = \max_{k=1,\dots,Q} n_k(A) = K_i - \sum_{j:A_j^{(i)} \cap A = \varnothing} 1 = \sum_{j:A_j^{(i)} \cap A \neq \varnothing} 1.$$

Then the following equalities are valid for any $A \neq \varnothing$ and $A \neq L$:

$$\underline{P}(A|s) = \frac{Z_1(A)}{K_i + s}, \quad \overline{P}(A|s) = \frac{Z_2(A) + s}{K_i + s}.$$

If $A = \varnothing$, then $\underline{P}(A|s) = \overline{P}(A|s) = 0$. If $A = L$, then $\underline{P}(A|s) = \overline{P}(A|s) = 1$.

Proposition 1 shows that the lower and upper probabilities of arbitrary events closely depend on the number K_i of observations and the hyperparameter s.

3.4 Additional interpretations of $\underline{P}(A|s)$ and $\overline{P}(A|s)$

3.4.1 $\underline{P}(A|s)$ and $\overline{P}(A|s)$ as the parametric extensions of belief and plausibility functions

Suppose K_i observations of an element $u \in L$ were made, each of which resulted in an imprecise (non-specific) measurement given by a set A_j of values. Let c_j denote the number of occurrences of the set $A_j \subseteq L$, and $\mathcal{P}(L)$ the set of all subsets of L (power set of L). Here $\sum_{j:A_j \neq \varnothing} c_j = K_i$. In the framework of *random set theory*[14, 15], a frequency function m, called *basic probability assignment*, can be defined as follows:

$$m : \mathcal{P}(L) \to [0,1], \quad m(\varnothing) = 1, \quad \sum_{A \in \mathcal{P}(L)} m(A) = 1.$$

This function can be also obtained as $m(A_j) = c_j/K_i$. Then the *belief* $Bel(A)$ and *plausibility* $Pl(A)$ *functions* [14, 15] of an event $A \subseteq L$ are defined as

$$Bel(A) = \sum_{j:A_j \subseteq A} m(A_j), \quad Pl(A) = \sum_{j:A_j \cap A \neq \varnothing} m(A_j).$$

It is noteworthy that the obtained probabilities $\underline{P}(A|s)$ and $\overline{P}(A|s)$ can be regarded as some parametric extensions of the belief $Bel(A)$ and plausibility $Pl(A)$ functions of A. According to [22, 23], we can write

$$\underline{P}(A|s) = \frac{K_i \cdot Bel(A)}{K_i + s}, \ \overline{P}(A|s) = \frac{K_i \cdot Pl(A) + s}{K_i + s}.$$

If $s = 0$, then $\underline{P}(A, 0) = Bel(A)$, $\overline{P}(A, 0) = Pl(A)$. However, by using the probabilistic interpretation of the belief and plausibility functions[24], we need to have a lot of judgments in order to compute the so-called *basic probability assignments* for events and this is just unrealistic in many reliability applications. At the same time, by increasing the hyperparameter s, we can make the results to be more cautious and the possible large imprecision of results reflects insufficiency of available information.

Moreover, $\underline{P}(A|s)$ and $\overline{P}(A|s)$ are again belief and plausibility functions with the basic probability assignment $m^*(A_j) = c_j/(K_i + s)$ for every A_j and the additional basic probability assignment $m^*(L) = s/(K_i + s)$, i.e., $\underline{P}(A|s)$ and $\overline{P}(A|s)$ can be obtained as standard belief and plausibility functions under condition that there are s additional (non-informative) observations $A_{K_i+1} = L$. If we denote $m(A_j) = c_j/K_i$, then $m^*(A_j) = m(A_j) \cdot K_i/(K_i + s)$ and

$$\underline{P}(A|s) = \sum_{j:A_j \subseteq A} m^*(A_j), \ \overline{P}(A|s) = m^*(L) + \sum_{j:A_j \cap A \neq \varnothing} m^*(A_j).$$

The above also follows from an interpretation of the hyperparameter s as the number of *hidden* observations[18].

Table 2: Lower and upper distribution functions by $s = 1$

r	A	$Z_1(\ A)$	$Z_2(\ A)$	$\underline{F}_i(r\|1)$	$\overline{F}_i(r\|1)$
0	$\{0,1,2,3\}$	5	5	1	1
1	$\{1,2,3\}$	4	5	0.667	1
2	$\{2,3\}$	2	2	0.333	0.5
3	$\{3\}$	1	1	0.166	0.333

3.4.2 Robust interpretation

The obtained probabilities can be also considered in the framework of robust ε-contaminated models[25]. As pointed out by Seidenfeld and Wasserman in the discussion of Walley's paper[18], the IDM has the same lower and upper probabilities as the ε-contaminated model (a class of probabilities which for fixed $\varepsilon \in (0,1)$ and $P(A)$ is the set $\{(1-\varepsilon)P(A)+\varepsilon Q(A) : Q \text{ is arbitrary}\}$). Here $P(A) = n(A)/K_i$ and $\varepsilon = s/(K_i+s)$, $n(A)$ is the number of observations of an event A. This implies that the probabilities $\underline{P}(A|s)$ and $\overline{P}(A|s)$ can be regarded as extended (contaminated) probabilities. On one hand, this representation of the probabilities has a clear explanation and allows us to control the bounds by changing the value of ε. On the other hand, it has a shortcoming. By obtaining new judgments or observations, the value of ε has to be changed and it is difficult to define rules for doing that. As pointed out by Walley[18], s must not depend on the number of judgments. This implies that the hyperparameter s remains without changes and the lower and upper probabilities depend only on K_i.

3.5 Probability distribution functions of the component states

By knowing the lower and upper probabilities of arbitrary subsets of states, the lower $\underline{F}_i(r|s)$ and upper $\overline{F}_i(r|s)$ bounds for the probability distribution function of the component states can be calculated by a certain value of s. In this case, we have to calculate the probabilities of subsets $A = \{r, r+1, ..., m\}$ for all $r = 0, ..., m$. It follows from Proposition 1 that

$$\underline{F}_i(r|s) = \frac{Z_1(\{r,...,m\})}{K_i+s} = \frac{\sum_{j:A_j^{(i)} \subseteq \{r,...,m\}} 1}{K_i+s}, \tag{1}$$

for $r \geq 1$ and $\underline{F}_i(r|s) = 1$ for $r = 0$. The upper bound is

$$\overline{F}_i(r|s) = \frac{Z_2(\{r,...m\})+s}{K_i+s} = \frac{K_i - \sum_{j:A_j^{(i)} \cap \{r,...,m\}=\varnothing} 1 + s}{K_i+s}. \tag{2}$$

For example, suppose that we have a 4-state component and five observations ($K_i = 5$) of states at time t: $A_1^{(i)} = \{1,2\}$, $A_2^{(i)} = \{1,2\}$, $A_3^{(i)} = \{0,1\}$, $A_4^{(i)} = \{3\}$, $A_5^{(i)} = \{2\}$. By using (1) and (2) we compute the probability distributions shown in Table 2.

Proposition 2 *Denote* $\underline{a}_j^{(i)} = \inf A_j^{(i)}$ *and* $\overline{a}_j^{(i)} = \sup A_j^{(i)}$. *The lower* $\underline{F}_i(r|s)$ *and upper* $\overline{F}_i(r|s)$ *distribution functions are step-wise functions with jumps at points from sets* $\mathcal{A} = \left\{\underline{a}_j^{(i)}, j = 1, ..., K_i\right\}$ *and* $\mathcal{B} = \left\{\overline{a}_j^{(i)}, j = 1, ..., K_i\right\}$, *respectively. Moreover, if* $s > 0$, *then* $\underline{F}_i(0|s) = 1$.

It follows from Proposition 2 that (1) and (2) can be rewritten as

$$\underline{F}_i(r|s) = \frac{\sum_{j:\underline{a}_j^{(i)} \geq r} 1}{K_i+s},\ \overline{F}_i(r|s) = \frac{K_i - \sum_{j:\overline{a}_j^{(i)} < r} 1 + s}{K_i+s} = \frac{\sum_{j:\overline{a}_j^{(i)} \geq r} 1 + s}{K_i+s}. \tag{3}$$

It is worth noticing that the above bounds do not depend on the number of states of components and are defined only intervals $A_j^{(i)}$. This interesting feature will be used by analyzing the CS systems.

Let us consider a case of total ignorance about states of the i-th component. Before making any observations, we have $K_i = 0$ and the set of intervals $A_j^{(i)}$ is empty. It follows from (3) in this case that the prior distribution function has the bounds $\underline{F}_i(r|s) = 0$ and $\overline{F}_i(r|s) = 1$ for all r and $s > 0$. When nothing is known about a component in advance, then it is natural that some "true" probability distribution may be arbitrary in the interval $[0, 1]$.

Let us consider a case when $K_i \to \infty$, i.e., we have infinitely many observations. If all observations are precise, i.e., $\underline{a}_j^{(i)} = \overline{a}_j^{(i)}$, then

$$\sum\nolimits_{j:\underline{a}_j^{(i)} \geq r} 1/K_i = n_i/K_i, \ \sum\nolimits_{j:\overline{a}_j^{(i)} < r} 1 = 1 - n_i/K_i.$$

Hence $\underline{F}_i(r|s) = \overline{F}_i(r|s)$. In this case, the finite hyperparameter s does not impact on the resulting reliability measures.

3.6 Updating the component reliability measures

Since Dirichlet distributions are conjugate priors (they generate Dirichlet posterior distributions), then the procedure of updating the component reliability measures is very simple. By observing a new event $A_k^{(i)}$ such that $\underline{a}_k^{(i)} = \inf A_k^{(i)} \geq r$, expressions (3) are updated as $\overline{F}_i^{\text{updated}}(r|s) = \overline{F}_i(r|s)$ and

$$\begin{aligned}\underline{F}_i^{\text{updated}}(r|s) &= \frac{1 + (K_i + s)\underline{F}_i(r|s)}{K_i + s + 1} \\ &= \underline{F}_i(r|s) + \frac{(\underline{F}_i(r|s))^{-1} - 1}{K_i + s + 1} \geq \underline{F}_i(r|s).\end{aligned}$$

If $\overline{a}_k^{(i)} = \sup A_k^{(i)} < r$, then $\underline{F}_i^{\text{updated}}(r|s) = \underline{F}_i(r|s)$ and

$$\overline{F}_i^{\text{updated}}(r|s) = \overline{F}_i(r|s) - \frac{(1 - \overline{F}_i(r|s))^{-1} - 1}{K_i + s + 1} \leq \overline{F}_i(r|s).$$

If $\underline{a}_k^{(i)} < r$ and $\overline{a}_k^{(i)} \geq r$, then $\overline{F}_i^{\text{updated}}(r|s) = \overline{F}_i(r|s)$ and $\underline{F}_i^{\text{updated}}(r|s) = \underline{F}_i(r|s)$. It can be seen from the above results that the imprecision of estimates is reduced after obtaining new observations.

4 Multi-state system reliability measures

Proposition 3 *Suppose that the k-th set Ψ_k of probability distribution functions of the random variable X_k defined on the sample space $\{0, ..., m\}$ is restricted by lower and upper distributions (3). Let $S(X)$ be a monotone non-decreasing function of n independent random variables $X = (X_1, ..., X_n)$ (structural function of a system). Denote $\underline{a}_0^{(i)} = 0$ and $\overline{a}_0^{(i)} = m$. Then the lower and upper bounds for the probability $F(r) = \Pr\{S(X) \geq r\}$ are determined as follows:*

$$\underline{F}(r|s) = \sum_V \prod_{i=1}^n \frac{\sum_{j \geq 0:\underline{a}_j^{(i)} = j_i} d_j}{K_i + s}, \quad \overline{F}(r|s) = \sum_V \prod_{i=1}^n \frac{\sum_{j \geq 0:\overline{a}_j^{(i)} = j_i} d_j}{K_i + s}. \tag{4}$$

Here

$$d_j = \begin{cases} s, & j = 0 \\ 1, & j \neq 0 \end{cases},$$

$$V = \{(j_1, ..., j_n) : j_i \in \{0, 1, ..., m\}, \ S(j_1, ..., j_n) \geq r\}.$$

It follows from Proposition 3 that the lower $\underline{F}(r|s)$ (upper $\overline{F}(r|s)$) bound for the probability distribution function of states of an arbitrary monotone MS system is defined by lower (upper) bounds for the probability distribution function of the component states. This is very important property which allows us to simplify calculations of the system reliability measures. Moreover, we do not need to deal with all points of the distribution functions, but only with bounds of available subsets of states. Therefore, the advantage of the approach increases when the number of observation is rather small. At the same time, the proposed extension of the IDM allows us to cope with insufficient statistical data about the component reliability.

It should be noted that the above algorithm can be simplified.

Corollary 1 *Let us order bounds $\underline{a}_j^{(i)}$ and $\overline{a}_j^{(i)}$ and unite identical lower and upper bounds, respectively. As a result, there are a set of different bounds $\underline{b}_k^{(i)}$, $k = 1, ..., N_i$, and a set of different bounds $\overline{b}_k^{(i)}$, $k = 1, ..., M_i$. Let $c_k^{(i)}$ denote the number of identical bounds $\underline{a}_k^{(i)}$ and $e_k^{(i)}$ denote the number of identical bounds $\overline{a}_k^{(i)}$ such that $\sum_{k=1}^{N_i} c_k^{(i)} = K_i$ and $\sum_{k=1}^{M_i} e_k^{(i)} = K_i$. At that, there hold $c_0^{(i)} = s$ and $e_0^{(i)} = s$. It follows from Proposition 3 that*

$$\underline{F}(r|s) = \sum_{Y} \prod_{i=1}^{n} \frac{c_{k_i}^{(i)}}{K_i + s}, \quad \overline{F}(r|s) = \sum_{Z} \prod_{i=1}^{n} \frac{e_{k_i}^{(i)}}{K_i + s}.$$

Here

$$Y = \{(\underline{b}_{k_1}^{(1)}, ..., \underline{b}_{k_n}^{(n)}) : S(\underline{b}_{k_1}^{(1)}, ..., \underline{b}_{k_n}^{(n)}) \geq r\},$$

$$Z = \{(\overline{b}_{k_1}^{(1)}, ..., \overline{b}_{k_n}^{(n)}) : S(\overline{b}_{k_1}^{(1)}, ..., \overline{b}_{k_n}^{(n)}) \geq r\}.$$

It can be seen from Corollary 1 that we do need to enumerate all states of components, but we can deal only with bounds of observations, i.e., we deal only with a restricted number of states coinciding with bounds of the observations. This is an important fact allowing us to simplify the computation procedure for reliability analysis of CS systems.

A flow chart of the computational algorithm to compute the lower bound $\underline{F}(r|s)$ for the probability distribution function of the system states is shown in Fig. 1. The upper bound $\overline{F}(r|s)$ can be computed in the same way if to replace $\underline{b}_j^{(i)}$ by $\overline{b}_j^{(i)}$, $c_k^{(i)}$ by $e_k^{(i)}$ and N_i by M_i.

4.1 Special cases

4.1.1 Series system

A series structure is such that $S(X) = \min(x_1, x_2, \ldots, x_n)$. It is known that the probability distribution function of the system states $F(r)$ is expressed through the probability distribution functions of the component states as $F(r) = F_1(r) \cdots F_n(r)$. It follows from (3) and Proposition 3 that

$$\underline{F}(r|s) = \prod_{i=1}^{n} \frac{\sum_{j:\underline{a}_j^{(i)} \geq r} 1}{K_i + s}, \quad \overline{F}(r|s) = \prod_{i=1}^{n} \frac{\sum_{j:\overline{a}_j^{(i)} \geq r} 1 + s}{K_i + s}.$$

If all data are precise, i.e., we have $c_j^{(i)}$ precise ($\underline{a}_j^{(i)} = \overline{a}_j^{(i)}$) observations of the j-th state for the i-th component, $j = 1, ..., m$, $i = 1, ..., n$, then

$$\underline{F}(r|s) = \prod_{i=1}^{n} \frac{\sum_{j=r}^{m} c_j^{(i)}}{K_i + s}, \quad \overline{F}(r|s) = \prod_{i=1}^{n} \frac{\sum_{j=r}^{m} c_j^{(i)} + s}{K_i + s}.$$

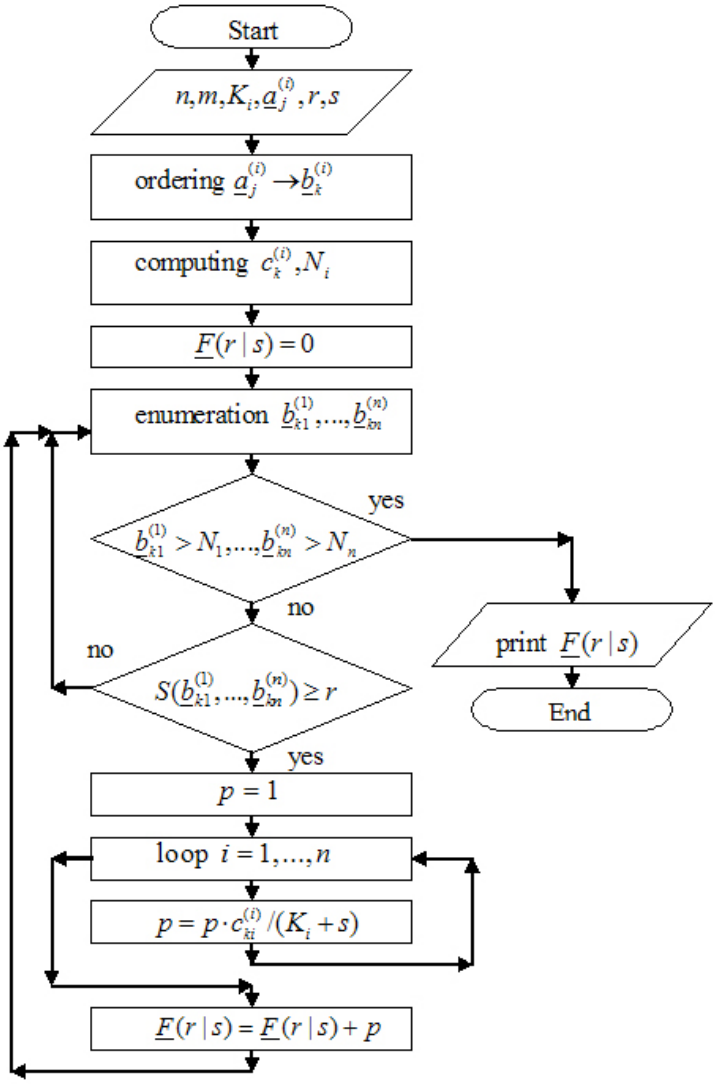

Figure 1: A flow chart to compute $\underline{F}(r|s)$

4.1.2 Parallel system

A parallel structure is such that $S(X) = \max(x_1, x_2, \ldots, x_n)$. It is known that the probability distribution function of the system states $F(r)$ is expressed through the probability distribution functions of the component states as $F(r) = 1 - (1 - F_1(r)) \cdots (1 - F_n(r))$. It follows from (3) and Proposition 3 that

$$\underline{F}(r|s) = 1 - \prod_{i=1}^{n} \frac{\sum_{j:\underline{a}_j^{(i)} < r} 1 + s}{K_i + s}, \quad \overline{F}(r|s) = 1 - \prod_{i=1}^{n} \frac{\sum_{j:\overline{a}_j^{(i)} < r} 1}{K_i + s}.$$

If all data are precise, i.e., we have $c_j^{(i)}$ precise observations of the j-th state for the i-th component, $j = 1, ..., m$, $i = 1, ..., n$, then

$$\underline{F}(r|s) = 1 - \prod_{i=1}^{n} \frac{\sum_{j=0}^{r-1} c_j^{(i)} + s}{K_i + s}, \quad \overline{F}(r|s) = 1 - \prod_{i=1}^{n} \frac{\sum_{j=0}^{r-1} c_j^{(i)}}{K_i + s}.$$

5 Reliability analysis of continuum-state systems

Now we consider a CS system whose components are in states from the interval $L = [0, T]$, $T \in \mathbf{R}^+$. Note that we can not simply extend the results obtained for MS systems on CS systems because the number of states is infinite and the Dirichlet model has infinitely many variables. However, it can be seen that the set of intervals $A_j^{(i)}$ about states of the i-th component divides the interval $[0, T]$ of all states of the component into a set of non-intersecting "smallest" intervals covering all states of the component. The term "smallest" means that there is total ignorance about states inside these intervals and we can not assign probability masses to some parts of the interval. How to find the smallest intervals? Let $\{\mathbf{k}\} = \{(k_1, ..., k_{K_i+1})\}$ be a set of all binary vectors consisting of $K_i + 1$ components such that $k_j \in \{0, 1\}$. For every vector $\mathbf{k}$, we define the interval $B_k \subseteq [0, T]$ $(k = 1, ..., 2^{K_i+1})$ as follows:

$$B_k = \left(\bigcap_{j:k_j=1} A_j^{(i)} \right) \bigcap \left(\bigcap_{j:k_j=0} \left(A_j^{(i)} \right)^c \right), \ k_j \in \mathbf{k}.$$

Here $A_{K_i+1}^{(i)} = [0, T]$ and $\left(A_j^{(i)} \right)^c$ is the complement of $A_j^{(i)}$. By uniting possible identical intervals B_k obtained in case of identical intervals $A_j^{(i)}$ and "empty" intervals obtained only from complements of $A_j^{(i)}$, we divide the interval $[0, T]$ into a set of M non-intersecting intervals B_k such that $B_1 \cup ... \cup B_M = \Omega$, $M \leq 2^{K_i+1}$. Moreover, every interval $A_j^{(i)}$ can be represented as the union of a finite number of intervals B_k. For example, suppose that we observe intervals of states: $A_1 = [6, 14]$, $A_2 = [4, 9]$, $A_3 = [2, 11]$, $A_4 = [4, 9]$ and $T = 100$. Then intervals B_k are of the form: $B_2 = [2, 4]$, $B_3 = [4, 6]$, $B_4 = [6, 9]$, $B_5 = [9, 11]$, $B_6 = [11, 14]$, and $B_1 \cup B_7 = [0, 2] \cup [14, 100]$.

In sum, the continuous set of states $[0, T]$ is reduced to a discrete sample space with M elements. It will be shown below that intervals B_k are not important for reliability analysis. We need them only for constructing the Dirichlet model.

Let us return to the urn model considered in Section 3. Every interval B_k now is associated with a urn. By constructing the set of multinomial model, we do not need to know the size of intervals B_k. Therefore, all the expressions (Proposition 1) proved for MS components are valid for CS systems.

It follows from (1) and (2) that the lower bound for the probability distribution functions of the i-th component states is

$$\underline{F}_i(r|s) = \frac{\sum_{j:A_j^{(i)} \subseteq [r,T]} 1}{K_i + s} = \frac{\sum_{j:\underline{a}_j^{(i)} \geq r} 1}{K_i + s},$$

for $r > 0$ and $\underline{F}_i(r|s) = 1$ for $r = 0$. The upper bound is

$$\overline{F}_i(r|s) = \frac{\sum_{j:A_j^{(i)} \cap [r,T] \neq \varnothing} 1 + s}{K_i + s} = \frac{\sum_{j:\overline{a}_j^{(i)} \geq r} 1 + s}{K_i + s},$$

for $r < T$ and $\underline{F}_i(r|s) = 0$ for $r = T$.

Since the lower and upper probability distribution functions are step-wise functions (see Proposition 2), then the corresponding density functions are the weighted sum of Dirac functions $\delta(x - x_i)$ which have unit area concentrated in the immediate vicinity of some points x_i, i.e.,

$$q_i(x) = \sum_{k=1}^{K_i+1} c_k \delta(x - x_k),$$

where $\sum_{k=1}^{K_i+1} c_k = 1$, $x_k \in [0, T]$.

Proposition 4 *Proposition 3 and Corollary 1 are valid under condition that the random variable X_k is defined on the sample space $[0, T]$.*

Proposition 4 implies that the reliability calculations for CS systems do not differ from the described algorithm for calculating the reliability measures of MS systems.

6 Numerical example

Let us analyze the offshore electrical power generation system considered by Natvig et al.[26]. The system consists of two generators and a control unit. Each of the generators can (if working properly) supply an oil rig with a sufficient amount of power, and hence only one generator is normally working while the other is in cold standby. If the operating generator experiences trouble, then the control unit is responsible for starting the spare generator. Each of the two generators can be in any of three states $\{0, 1, 2\}$. The control unit can also be in any of three states: {0 (halts the operating generator without starting the spare), 1 (fails to start the spare generator), 2 (operating properly). The amount of power that can be supplied to the platform is represented by a MS structure function

$$S(x_1, x_2, x_3) = \mathbf{1}_{x_1>0} \cdot \min(x_2 + x_3 \cdot \mathbf{1}_{x_2=2}, 2).$$

where x_1 denotes states of the control unit, x_2 and x_3 denote states of two generators, and $\mathbf{1}$ is the indicator function.

Suppose that preliminary unreliable observations of the components give the following interval-valued states:

1. the control unit: 4 times {1}, 3 times {1,2}.
2. generators: 2 times {0}, 1 time {1,2}, 1 time {0,1}.

The initial estimates can be formalized as follows:

1. The control unit: $\underline{a}_j^{(1)} = \overline{a}_j^{(1)} = 1$, $j = 1, ..., 4$, $\underline{a}_j^{(1)} = 1$, $\overline{a}_j^{(1)} = 2$, $j = 5, 6, 7$.
2. Generators ($i = 2, 3$): $\underline{a}_j^{(i)} = \overline{a}_j^{(i)} = 0$, $j = 1, 2$, $\underline{a}_3^{(i)} = 1$, $\overline{a}_3^{(i)} = 2$, $\underline{a}_4^{(i)} = 0$, $\overline{a}_4^{(i)} = 1$.

We additionally write $\underline{a}_0^{(i)} = 0$, $\overline{a}_0^{(i)} = 2$, $i = 1, 2, 3$. Let $s = 1$. At first, we find the probability distribution functions of the component states:

1. The control unit:

$$\underline{F}_1(0|1) = 1,\ \overline{F}_1(0|1) = 1,$$
$$\underline{F}_1(1|1) = 7/8,\ \overline{F}_1(1|1) = 1,$$
$$\underline{F}_1(2|1) = 1/8,\ \overline{F}_1(2|1) = 4/8.$$

2. Generators ($i = 2, 3$):

$$\underline{F}_i(0|1) = 1,\ \overline{F}_i(0|1) = 1,$$
$$\underline{F}_i(1|1) = 1/5,\ \overline{F}_i(1|1) = 3/5,$$
$$\underline{F}_i(2|1) = 0,\ \overline{F}_i(2|1) = 2/5.$$

By ordering the lower $\underline{a}_j^{(i)}$ and upper $\overline{a}_j^{(i)}$ bounds, we get the sets

$$\underline{b}_0^{(1)} = 0,\ \underline{b}_1^{(1)} = 1,\ \overline{b}_0^{(1)} = 2,\ \overline{b}_1^{(1)} = 1,\ \overline{b}_2^{(1)} = 2,$$
$$c_0^{(1)} = s,\ c_1^{(1)} = 7,\ e_0^{(1)} = s,\ e_1^{(1)} = 4,\ e_2^{(1)} = 3,$$

$$\underline{b}_0^{(i)} = 0,\ \underline{b}_1^{(i)} = 0,\ \underline{b}_2^{(i)} = 1,\ \overline{b}_0^{(i)} = 2,\ \overline{b}_1^{(i)} = 0,\ \overline{b}_2^{(i)} = 1,\ \overline{b}_3^{(i)} = 2,\ i = 1, 2,$$
$$c_0^{(i)} = s,\ c_1^{(i)} = 3,\ c_1^{(i)} = 1,\ e_0^{(i)} = s,\ e_1^{(i)} = 2,\ e_2^{(i)} = 1,\ e_3^{(i)} = 1,\ i = 1, 2.$$

By using Corollary 1 and the algorithm depicted in Fig. 1, we get the probability distribution functions of the system states

$$\underline{F}(0|1) = 1,\ \overline{F}(0|1) = 1,$$
$$\underline{F}(1|1) = 0.175,\ \overline{F}(1|1) = 0.6,$$
$$\underline{F}(2|1) = 0,\ \overline{F}(2|1) = 0.4.$$

Note that if we take $s = 0$, then

$$\underline{F}(0|0) = 1,\ \overline{F}(0|0) = 1,$$
$$\underline{F}(1|0) = 0.25,\ \overline{F}(1|0) = 0.5,$$
$$\underline{F}(2|0) = 0,\ \overline{F}(2|0) = 0.25.$$

It can be seen from the above results that the reliability measures obtained by $s = 1$ are more cautious than the same measures by $s = 0$. At the same time, they are more imprecise.

7 Conclusion

The simple expressions for lower and upper probability distribution functions of states of MS and CS systems have been proposed in the paper under condition that initial data about reliability of components are represented in the form of interval-valued observations or measurements. The obtained reliability measures can be also interpreted in terms of random set theory and robust statistical models. The main virtue of the proposed expressions and the algorithm for their calculation is that they can be applied in the case when the number of observations or measurements is rather small. This is provided by varying the hyperparameter s of the IDM. The second virtue of the proposed algorithm for computing the reliability measures is that it can be applied to both MS and CS systems without an additional adaptation.

It should be noted that the expressions and the algorithm for computing probability distribution functions of the system states have been considered in the paper. However, different reliability measures can be simply obtained from these measures.

8 Appendix

Proof of Proposition 1: Note that $\inf_{\alpha \in S(1,m+1)} \alpha(A)$ is achieved at $\alpha(A) = 0$ and $\sup_{\alpha \in S(1,m+1)} \alpha(A)$ is achieved at $\alpha(A) = 1$ except a case when $A = L$. If $A = L$, then $\alpha(A) = 1$ for the minimum and maximum. In order to find the minimum and maximum of $n_k(A)$ we consider three types of subsets $A_{j1}^{(i)}$, $A_{j2}^{(i)}$, $A_{j3}^{(i)}$ produced by observations such that $A_{j1}^{(i)} \subseteq A$, $A_{j2}^{(i)} \cap A = \varnothing$, $A_{j3}^{(i)} \cap A \neq \varnothing$ and $A_{j3}^{(i)} \not\subseteq A$. It is obvious that balls corresponding to sets $A_{j1}^{(i)}$ belong to the set A and $n_k(A)$ can not be less than $\sum_{j:A_{j1}^{(i)} \subseteq A} 1$. On the other hand, balls corresponding to sets $A_{j2}^{(i)}$ do not belong to A. This implies that $n_k(A)$ can not be greater than $K_i - \sum_{j:A_{j2}^{(i)} \cap A = \varnothing} 1$. Balls corresponding to $A_{j3}^{(i)}$ may belong to A, but it is not necessary. Therefore, $\min n_k(A) = \sum_{j:A_j^{(i)} \subseteq A} 1$ and $\max n_k(A) = K_i - \sum_{j:A_j^{(i)} \cap A = \varnothing} 1$, as was to be proved. ■

Proof of Proposition 2: Note that $A_j^{(i)} \subseteq \{r, ..., m\}$ if $\underline{a}_j^{(i)} \geq r$. Suppose that the points $r \notin \mathcal{A}$ and $r+1 \notin \mathcal{A}$. This implies that if $A_j^{(i)} \subseteq \{r, ..., m\}$, then $\underline{a}_j^{(i)} > r$. However, $\underline{a}_j^{(i)} \neq r+1$ because $r+1 \notin \mathcal{A}$. Consequently, $\underline{a}_j^{(i)} > r+1$ and $A_j^{(i)} \subseteq \{r+1, ..., m\}$. It follows from (1) that $\underline{F}_i(r|s) - \underline{F}_i(r+1|s) = 0$. Suppose now that $r \in \mathcal{A}$ and $r = \underline{a}_j^{(i)}$. This implies that $A_j^{(i)} \subseteq \{r, ..., m\}$ and $A_j^{(i)} \subsetneq \{r+1, ..., m\}$. Hence $\underline{F}_i(r|s) \geq \underline{F}_i(r|s)$. Therefore, jumps of the lower distribution function are at points $\underline{a}_j^{(i)}$. The proof for the upper distribution function is similar if to take into account that $A_j^{(i)} \cap \{r, ..., m\} = \varnothing$ if $\overline{a}_j^{(i)} < r$. ■

Proof of Proposition 3: Denote $q_k(x) = F_k(i) - F_k(i+1)$, where $F_k \in \Psi_k$. Then

$$\Pr\{S(X) \geq r\} = \sum_{S(X) \geq r} \prod_{k=1}^{n} q_k(x_k).$$

Since $S(X)$ is non-decreasing with every variable, then we can rewrite the above equality as follows:

$$\Pr\{S(X) \geq r\} = \sum_{z=0}^{m} q_l(z) Q(z),$$

where

$$Q(z) = \sum_{S(X_z) \geq r} \prod_{k=1, k \neq l}^{n} q_k(x_k), \; S(X_z) = S(x_1, ..., z_l, ..., x_n).$$

Since $q_l(z) = F_l(z) - F_l(z+1)$, then

$$\begin{aligned} \Pr\{S(X) \geq r\} &= \sum_{z=0}^{m} \left(F_l(z) - F_l(z+1)\right) Q(z) \\ &= D + \sum_{z=0}^{m-1} F_l(z+1) \left[Q(z+1) - Q(z)\right], \end{aligned}$$

where $D = F_k(0)Q(0) - F_k(m+1)Q(m)$ is some finite real number.

Since $S(X)$ is non-decreasing, then

$$S(x_1, ..., z_1, ..., x_r) \geq S(x_1, ..., z_2, ..., x_r) \geq r$$

for $z_1 \geq z_2$, and the number of terms of the sum for computing Q increases as z increases. This implies that $Q(z)$ is non-decreasing with z and $Q(z+1) - Q(z) \geq 0$. Consequently, in order to minimize (maximize) $\Pr\{S(X) \geq r\}$, we have to take $F_l(z)$ as smaller (larger) as possible, i.e., the minimum (maximum) of $\Pr\{S(X) \geq r\}$ is achieved at $F_l = \underline{F}_l$ ($F_l = \overline{F}_l$). However, it follows from Proposition 2 that the lower (upper) distribution function is completely defined by $\underline{a}_j^{(i)}$ ($\overline{a}_j^{(i)}$). Then

$$\min \Pr\{S(X) \geq r\} = \sum_{V} \prod_{k=1}^{n} \underline{q}_k(j_k|s), \; \max \Pr\{S(X) \geq r\} = \sum_{V} \prod_{k=1}^{n} \overline{q}_k(j_k|s).$$

Here $\underline{q}_k(j|s) = \underline{F}_k(j|s) - \underline{F}_k(j+1|s)$, $\overline{q}_k(j|s) = \overline{F}_k(j|s) - \overline{F}_k(j+1|s)$, $\underline{F}_k(m+1|s) = \overline{F}_k(m+1|s) = 0$, $\underline{F}_k(0|s) = 1$. It follows from (3) that

$$\underline{F}_k(r|s) - \underline{F}_k(r+1|s) = \sum_{j:\underline{a}_j^{(i)} = r} 1/(K_i + s),$$

$$\overline{F}_k(r|s) - \overline{F}_k(r+1|s) = \sum_{j:\overline{a}_j^{(i)} = r} 1/(K_i + s).$$

If $r = 0$, then

$$\underline{q}_k(0|s) = \underline{F}_k(0|s) - \underline{F}_k(1|s) = s/(K_i + s) + \sum_{j:\underline{a}_j^{(i)}=0} 1/(K_i + s).$$

If $r = m$, then

$$\overline{q}_k(m|s) = \overline{F}_k(m|s) - 0 = s/(K_i + s) + \sum_{j:\overline{a}_j^{(i)}=m} 1/(K_i + s).$$

This completes the proof. ■

Proof of Corollary 1: Suppose that for some value j_k in the vector $(j_1, ..., j_n)$ from V there is no $\underline{a}_j^{(k)}$ such that $\underline{a}_j^{(k)} = j_k$. This implies that the product in (4) is 0. Consequently, non-zero products take place if there exist $\underline{a}_j^{(k)}$ such that $\underline{a}_j^{(k)} = j_k$ for all $k = 1, ..., n$. This implies that the set V can be reduced to the set Y. The following proof is obvious. ■

Proof of Proposition 4: The proof is similar to the proof of Proposition 3 except the following. Let $q_k(x) = \mathrm{d}F_k(x)/\mathrm{d}x$. Then

$$\Pr\{S(X) \geq r\} = \int_{S(X)\geq r} \cdots \int \prod_{k=1}^{n} q_k(x_k)\mathrm{d}X.$$

Hence

$$\Pr\{S(X) \geq r\} = \int_0^T q_l(z)Q(z)\mathrm{d}z,$$

where

$$Q(z) = \int_{S(x_1,...,z,...,x_r)\geq r} \cdots \int \prod_{k=1,k\neq l}^{n} q_k(x_k)\mathrm{d}X.$$

By integrating by parts, we get

$$\Pr\{S(X) \geq r\} = A - \int_0^T F_t(z)\frac{\mathrm{d}Q(z)}{\mathrm{d}z}\mathrm{d}z.$$

Here $A = F_t(\infty)Q(\infty) - F_t(-\infty)Q(-\infty)$ is some finite real number. $Q(z)$ is non-decreasing with z and $\mathrm{d}Q(z)/\mathrm{d}z \geq 0$. Since the density functions $q_k(x_k)$ are the weighted sum of Dirac functions, then the following proof is similar to the proof of Proposition 3 and Corollary 1. ■

References

[1] A.M. Abouammoh and M.A. Al-Kadi, Multistate coherent systems of order k, *Microelectronics and Reliability* **35** (1995) 1415–1421.

[2] R.E. Barlow and A.S. Wu, Coherent systems with multistate components, *Math. Ops. Res.* **3** (1978) 275–281.

[3] W.S. Griffith, Multistate reliability models, *J. Appl. Prob.* **17** (1980) 735–744.

[4] J. Montero, J. Tejada, and J. Yanez, General structure functions, *Kybernetes* **23** (1994) 10–19.

[5] G. Levitin and A. Lisnianski, *Multi-state system reliability. Assessment, optimization and applications* (World Scientific, Singapore, New Jersey, 2003).

[6] P. Walley, *Statistical Reasoning with Imprecise Probabilities* (Chapman and Hall, London, 1991).

[7] V. P. Kuznetsov, *Interval Statistical Models* (Radio and Communication, Moscow, 1991, in Russian).

[8] K. Weichselberger, *Elementare Grundbegriffe einer allgemeineren Wahrscheinlichkeitsrechnung, vol. I Intervallwahrscheinlichkeit als umfassendes Konzept* (Physika, Heidelberg, 2001).

[9] L.V. Utkin and S.V. Gurov, Imprecise reliability of general structures, *Knowledge and Information Systems* **1** (1999) 459–480.

[10] L.V. Utkin and I.O. Kozine, A reliability model of multi-state units under partial information, eds. H. Langseth and B. Lindqvist, in *Proc. of the Third Int. Conf. on Mathematical Methods in Reliability (Methodology and Practice)*, pp. 643–646 (Trondheim, Norway, June 2002, NTNU).

[11] I.O. Kozine and L.V. Utkin, Variety of judgements admitted in imprecise statistical reasoning, *Risk, Decision and Policy* **8** (2003) 111–120.

[12] L.V. Utkin and S.V. Gurov, New reliability models based on imprecise probabilities, ed. C. Hsu, in *Advanced Signal Processing Technology*, chap. 6, pp. 110–139 (World Scientific, 2001).

[13] B. Lindqvist and H. Langseth, Uncertainty bounds for a monotone multistate system, *Probability in the Engineering and Informational Sciences* **12** (1998) 239–260.

[14] A.P. Dempster, Upper and lower probabilities induced by a multi-valued mapping. *Ann, Math. Stat.* **38** (1967) 325–339.

[15] G. Shafer, *A Mathematical Theory of Evidence* (Princeton University Press, 1976).

[16] R.G. Almond, Models for incomplete failure data. Statistical Science Research Report 9, Statistical Sciences, Inc., Seattle, May 1992.

[17] P. Limbourg and D. Germann, Reliability assessment and optimization under uncertainty in the Dempster-Shafer framework, in *Proc. of the 27th European Safety, Reliability and Data Association Seminar on Assembling Evidence of Reliability. The Reliability Case as a Decision Support Tool*, pp. 77–89 (Glasgow, UK, November 2004).

[18] P. Walley, Inferences from multinomial data Learning about a bag of marbles, *Journal of the Royal Statistical Society, Series B*, **58** (1996) 3–57.

[19] M. DeGroot, *Optimal Statistical Decisions.* McGraw-Hill, New York, 1970.

[20] F.P.A. Coolen, An imprecise Dirichlet model for Bayesian analysis of failure data including right-censored observations, *Reliability Engineering and System Safety* **56** (1997) 61–68.

[21] M. Smithson, T. Bartos, and K. Takemura, Human judgment under sample space ignorance, eds. G. de Cooman, F.G. Cozman, S. Moral, and P. Walley, in *Proc. of the First International Symposium on Imprecise Probabilities and Their Applications*, pp. 324–332 (Zwijnaarde, Belgium, June 1999).

[22] L.V. Utkin, Belief functions and the imprecise Dirichlet model, eds. I. Batyrshin, J. Kacprzyk, and L. Sheremetov, in *Proc. of the Int. Conf. on Fuzzy Sets and Soft Computing in Economics and Finance*, vol.1, pp. 178–185 (St.Petersburg, Russia, June 2004, Mexican Petroleum Institute).

[23] L.V. Utkin. Probabilities of judgments provided by unknown experts by using the imprecise Dirichlet model. *Risk, Decision and Policy*, 9(4):391 – 400, 2004.

[24] J.Y. Halpern and R. Fagin, Two views of belief Belief as generalized probability and belief as evidence, *Artificial Intelligence* **54** (1992) 275–317.

[25] P.J. Huber, *Robust Statistics* (Wiley, New York, 1981).

[26] B. Natvig, S. Soermo, A.T. Holen, and G. Hogasen, Multistate reliability theory - a case study, *Adv. Appl. Prob.* **18** (1986) 577-579.