CAUTIOUS ANALYSIS OF PROJECT RISKS BY INTERVAL-VALUED INITIAL DATA

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Abstract

One of the most common performance measures in selection and management of projects is the Net Present Value (NPV). In the paper, we study a case when initial data about the NPV parameters (cash flows and the discount rate) are represented in the form of intervals supplied by experts. A method for computing the NPV based on using random set theory is proposed and three conditions of independence of the parameters are taken into account. Moreover, the imprecise Dirichlet model for obtaining more cautious bounds for the NPV is considered. Numerical examples illustrate the proposed approach for computing the NPV.

Keywords: net present value, random set theory, Dirichlet distribution, independence, interval-valued data, expert judgments

1 Introduction

The purpose of project risk management is to minimize the risks of not achieving the objectives of the project. Risks arise because of uncertainty about the future. One of the most common performance measures in selection and management of projects is the *Net Present Value* (NPV). To measure the NPV of a project, the relevant project cash flows are specified, and the time value of money is taken into account by discounting future cash flows by the required rate of return or the discount rate. In its simplest form, the NPV is determined as follows:

$$\mathrm{NPV} = V_0 + \sum_{k=1}^{T} \frac{V_k}{(1+r)^k}, \tag{1}$$

where V_0 is the *cash flow*[1] at year $k = 0$; r is the *discount rate*; V_k is equal to the cash flow at year k, $k = 1, ..., T$; T is the time period considered (in years or other time periods).

If a project has a positive NPV, then it is generally worth pursuing, in the absence of risk. The justification for this is that the company could borrow the necessary funds for investment, at the appropriate corporate cost of capital, the discount rate, and the project's returns would exceed the borrowing costs.

Various methods for quantitative risk assessments of projects by means of the NPV have been developed and analyzed in the literature [1, 2, 3] taking into account the fact that projects, by their nature, are unique and usually complex. The most methods can conditionally be divided into several types depending on uncertainty modelling of the parameters V_k, $k = 1, ..., T$, r of the NPV.

The first type of methods presupposes that all parameters of a project are precisely known. These deterministic methods are simplest, but they do not take into account the uncertainty of parameters.

Another part of methods take into account the fact that every parameter of a project is interval-valued. In this case, the standard procedures of interval analysis can be used for computing the interval-valued NPV.

[1] The cash flow V_0 is replaced sometimes by the investment cost I such that $I = -V_0$.

These methods can be regarded as extensions of the deterministic methods. However, by dealing with intervals in the framework of standard interval analysis, it is assumed that the "true" value of a project parameter belongs to the corresponding interval with probability 1. This is too strong assumption that does not hold in real problems.

The third type of methods considers every parameter of a project as a random variable governed by some precise probability distribution. These methods are well founded because they are based on the classical probability theory. However, they have the following shortcomings. Firstly, every project is unique by its nature and, therefore, statistical data obtained from analogous projects are not always justified and may lead to major errors. Secondly, one of the main sources of data for the project risk management is expert judgments, which are usually imprecise and unreliable due to the limited precision of human assessments. As a result, it is difficult to find a unique probability distribution fitted for the expert judgments. Thirdly, the choice of standard probability distributions corresponding to statistical data is carried out by a decision maker and is rather subjective in order to completely believe in it. Fourthly, these methods are computationally hard and mainly use approximate solutions. An example of the approximation is replacing the probability distributions by expectations of the NPV parameters and, sometimes, by their variances [1]. This approach is efficient enough, but the first moments of the random parameters are much less informative in comparison with the probability distributions.

The fourth type of methods [4] is based on using fuzzy set theory or possibility theory [5, 6], i.e., every parameter of a project is viewed as a fuzzy variable with a given possibility distribution or a membership function. However, these methods also have shortcomings. The choice of a form of possibility distributions is often carried out to simplify calculations, for instance, the well-known triangular forms. This might lead to incorrect results especially when the efficiency of a project is questionable and the NPV is close to 0. Secondly, the operations "min" and "max" used in fuzzy calculations may be ambiguous and are defined by independence conditions [7]. There exist other possible operations whose usage may give quite different final results. Moreover, the operations "min" and "max" lead to the large imprecision of results when the value of T is rather large and it is difficult to make decision in this case. The above shortcomings do not mean that the methods based on using fuzzy set theory should not be used in selection and management of projects. However, their features have to be confirmed before usage.

In the paper, we study a case when initial data about the NPV parameters (cash flows and the discount rate) are represented in the form of intervals supplied by experts. In order to processing this information, we propose a method for computing the NPV based on using random set theory [8, 9]. Moreover, we apply the imprecise Dirichlet model [10] for obtaining more cautious bounds for the NPV. At that, we take into account conditions of independence of the parameters as random variables. A question of using the possible information about quality of experts is also investigated in the paper. Numerical examples illustrate the proposed approach for computing the NPV.

The paper is organized as follows. The formal problem statement is given in Section 2. A short description of belief functions and main definitions of random sets are given in Section 3. The detailed studying of optimization problems for computing some characteristics of the NPV under conditions of random set independence, unknown interaction and strong independence of the NPV parameters can be found in Section 4. Section 5 considers a method for analyzing NPV by taking into account the expert quality. An extension of the obtained results by using the imprecise Dirichlet model is studied in Section 6.

2 Problem statement

Suppose that the available information about every parameter of the NPV is represented in the form of intervals such that there are c_{ki} intervals $\mathbf{V}_{ki} = [V_{ki}^{\mathrm{L}}, V_{ki}^{\mathrm{U}}]$, $i = 1, ..., n_k$, of the cash flow[2] V_k at year k, $k = 1, ..., T$, and d_j intervals $\mathbf{r}_j = [r_j^{\mathrm{L}}, r_j^{\mathrm{U}}]$, $j = 1, ..., m$, of the discount rate r. Here n_k and m are the numbers of different intervals of V_k and r, respectively. The total number of intervals concerning the k-th cash flow is $N_k = \sum_{i=1}^{n_k} c_{ki}$. In particular, if $c_{ki} = 1$ for all $i = 1, ..., n_k$, then $N_k = n_k$. The total number of intervals concerning the discount rate is $M = \sum_{j=1}^{m} d_j$. In particular, if $d_j = 1$ for all $j = 1, ..., m$, then $M = m$. By letters L and U, we denote lower and upper bounds of intervals, respectively.

[2]We assume that the value V_0 is precisely known.

Example 1 *Suppose we have four intervals* $[-4,-3]$*,* $[-4,-3]$*,* $[-4,-3]$*,* $[-1.95,1]$ *of the cash flow* V_1 *at the first year, one interval* $[2.4,7]$ *of the cash flow* V_2 *at the second year, two intervals* $[0.1,0.2]$*,* $[0.15,0.25]$ *of the discount rate* r*, and precise value* 0 *of* V_0 *(the investment cost is* 0*). Here* $T=2$*. The above can formally be written as follows:*

$$\mathbf{V}_{11}=[-4,-3],\ c_{11}=3,\ \mathbf{V}_{12}=[-1.95,1],\ c_{12}=1,\ n_1=2,\ N_1=4,$$

$$\mathbf{V}_{21}=[2.4,7],\ c_{21}=1,\ n_2=1,\ N_2=1,$$

$$\mathbf{r}_1=[0.1,0.2],\ d_1=1,\ \mathbf{r}_2=[0.15,0.25],\ d_2=1,\ m=2,\ M=2.$$

If we assume that every parameter of the NPV is a random variable with unknown probability distribution, then the NPV is also a random variable. This implies that possible measures of interest are the expectation $\mathbb{E}$NPV of the NPV and the probability $\Pr(\text{NPV}\geq 0)$ that the NPV exceeds the value 0[3]. Without loss of generality, we assume for simplicity that V_0 is precisely known.

3 Belief functions

Let U be a universal set under interest, usually referred to in evidence theory as the *frame of discernment*. Suppose N observations were made of an element $u\in U$, each of which resulted in an imprecise (non-specific) measurement given by a set A of values. Let c_i denote the number of occurrences of the set $A_i\subseteq U$, and $\mathcal{P}(U)$ the set of all subsets of U (power set of U). A frequency function m, called *basic probability assignment* (BPA), can be defined such that [8, 9]:

$$m:\mathcal{P}(U)\rightarrow[0,1],$$

$$m(\varnothing)=1,\ \sum_{A\in\mathcal{P}(U)}m(A)=1.$$

Note that the domain of BPA, $\mathcal{P}(U)$, is different from the domain of a probability density function, which is U. According to [8], this function can be obtained as follows:

$$m(A_i)=c_i/N. \tag{2}$$

If $m(A_i)>0$, i.e. A_i has occurred at least once, then A_i is called a *focal element*.

According to [9], the *belief* $Bel(A)$ and *plausibility* $Pl(A)$ measures of an event $A\subseteq\Omega$ can be defined as

$$Bel(A)=\sum_{A_i:A_i\subseteq A}m(A_i),\quad Pl(A)=\sum_{A_i:A_i\cap A\neq\varnothing}m(A_i). \tag{3}$$

As pointed out in [11], a belief function can formally be defined as a function satisfying axioms which can be viewed as a weakening of the Kolmogorov axioms that characterize probability functions. Therefore, it seems reasonable to understand a belief function as a generalized probability function [8] and the belief $Bel(A)$ and plausibility $Pl(A)$ measures can be regarded as lower and upper bounds for the probability of A, i.e., $Bel(A)\leq\Pr(A)\leq Pl(A)$.

Let us explain the belief and plausibility functions in terms of a *multivalued sampling process*. Consider a probability measure $P(\omega)$ defined on a universal set Ω (which can be thought of as the set of our observations) related to U (the set of the values of our measurements) through a multivalued mapping $G:\Omega\rightarrow\mathcal{P}(U)$. Then the BPA is [8]:

$$m(A_i)=P(\omega_i)=c_i/N,\ \omega_i\in\Omega.$$

This multivalued mapping expresses the imprecision of the measurement experienced during each observation, i.e., our inability to attach a single number to each observation. So, for each set $A_i\in\mathcal{P}(U)$, the value $m(A_i)$ expresses the probability of $\omega_i=G^{-1}(A_i)$ $(\omega_i\in\Omega)$. A *random set* is the pair $(\mathcal{F},m)$, where $\mathcal{F}$ is the family of all N focal elements.

[3]Generally, probabilities that the NPV exceeds arbitrary values may be of interest. However, we consider only $\Pr(\text{NPV}\geq 0)$ for simplicity because the non-zero value can always be taken into account in the parameter V_0.

Let A be a subset of U. If we define X_* as the subset of Ω whose elements must lead to A: $X_* = \{\omega \in \Omega : G(\omega) \subseteq A\}$, then the lower probability of A, according to Dempster's principle of inductive reasoning, is defined by $\underline{P}(A) = Bel(A) = P(X_*)$. If we define X^* as the subset of Ω whose elements may lead to A: $X^* = \{\omega \in \Omega : G(\omega) \cap A \neq \varnothing\}$, then the upper probability of A is given by $\overline{P}(A) = Pl(A) = P(X^*)$.

Suppose that U is the real line restricted by some values $\inf U$ and $\sup U$. Then we can define lower and upper cumulative probability distribution functions of a random variable X, about which we have data in the form of intervals A_i, $i = 1, ..., n$,

$$\underline{F}(x) = \underline{P}(\{u \leq x\}) = \sum_{i:\sup A_i \leq x} m(A_i) = N^{-1} \sum_{i:\sup A_i \leq x} c_i,$$

$$\overline{F}(x) = \overline{P}(\{u \leq x\}) = \sum_{i:\inf A_i \leq x} m(A_i) = N^{-1} \sum_{i:\inf A_i \leq x} c_i.$$

The distribution functions are an envelope of all the possible cumulative distribution functions compatible with the data and they allow us to calculate the lower and upper expectations of X. The lower and upper probability distributions can be considered in a framework of the so-called p-boxes [12]. A probability box (p-box) is a class of distribution functions delimited by some upper and lower bounds which collectively represent the epistemic uncertainty about the distribution function of a random variable. As indicated in [12], p-boxes are a somewhat coarser way to describe uncertainty than are Dempster-Shafer structures on the real line. Every Dempster-Shafer structure specifies a unique p-box and every p-box specifies an equivalence class of Dempster-Shafer structures.

4 NPV in the framework of random sets

By returning to Section 2 and viewing the parameters of the NPV as random variables, it can be seen that these parameters can be analyzed in the framework of random set theory. That is, the available information about the cash flow V_k at year k can be rewritten through BPAs[4] $m(\mathbf{V}_{ki}) = c_{ki}/N_k$, $i = 1, ..., n_k$. The similar BPAs are written for the discount rate $m(\mathbf{r}_j) = d_j/M$, $j = 1, ..., m$. Denote the set of focal elements (intervals) related to the k-th cash flow by $\mathcal{V}_k$, i.e.,

$$\mathcal{V}_k = \{\mathbf{V}_{ki},\ i = 1, ..., n_k\}, k = 1, ..., T, \tag{4}$$

and the set of focal elements related to the discount rate by $\mathcal{R}_k$, i.e.,

$$\mathcal{R} = \{\mathbf{r}_j,\ i = 1, ..., m\}. \tag{5}$$

Since the NPV is a multivariate function of random variables V_k, $k = 1, ..., T$, and r, then crucial points in the following analysis are conditions of independence, types of independence and monotonicity of the NPV.

When only partial information representing by sets of probability measures is available about random variables of a multivariate function, different types of independence [14, 15] can be considered. We do not stress on all the types of independence, but pick out three the most used types: *random set independence*, *unknown interaction* and *strong independence*.

The NPV is monotonically increasing with respect to V_k, $k = 1, ..., T$. However, it follows from the first derivative of the NPV

$$\frac{\partial NPV}{\partial r} = -\sum_{k=1}^{T} \frac{kV_k}{(1 + r)^{k+1}}$$

that the NPV is monotonically decreasing with respect to r if the cash flows are non-negative, i.e., $V_k \geq 0$ for all $k = 1, ..., T$. Moreover, the NPV is monotone if the discount rate is deterministic, i.e., r is a certain value. Generally, the NPV is the non-monotone function of r.

[4] It is assumed here that all intervals concerning one parameter of the NPV are obtained from one source. For simplicity, we do not consider the various combination rules [13] that can be applied to initial data obtained from different independent sources because every rule finally provides a set of BPAs which in principle do not differ from those of the case of one source.

4.1 Random set independence

Suppose that there is a function of two variables $x_1 \in U_1$ and $x_2 \in U_2$ and we know BPAs $m_1(A_1)$ and $m_2(A_2)$ of events $A_1 \subseteq U_1$ and $A_2 \subseteq U_2$, respectively. We say that there is random set independence [14, 8] if the joint BPA is defined by $m(A_1 \times A_2) = m_1(A_1)m_2(A_2)$. This definition of independence can be justified in terms of the multivalued sampling process under the following assumptions [14]: (a) there are two random experiments with possibility spaces Ω_1 and Ω_2, each of which is modeled by a known probability distribution; (b) each space Ω_i is related to U_i through a multivalued mapping $G_i : \Omega_i \to \mathcal{P}(U_i)$; (c) the probability distribution on Ω_i induces the BPA m_i on U_i through the multivalued mapping G_i; (d) the probability distributions on Ω_1 and Ω_2 are stochastically independent; (e) we know nothing about the interaction between the mechanisms for selecting the outcomes from U_1 and U_2.

Denote the set of indices

$$\mathcal{J} = \{(i_1, ..., i_T, j) : \ i_k = 1, ..., n_k, \ k = 1, ..., T, \ j = 1, ..., m\}$$

and the product of intervals

$$\mathbf{\Psi}(i_1, ..., i_T, j) = \mathbf{V}_{1i_1} \times ... \times \mathbf{V}_{Ti_T} \times \mathbf{r}_j \subseteq \mathbb{R}^{T+1}.$$

If there is random set independence, then in terms of the NPV computation problem the upper probability of an event $A \subseteq \mathbb{R}^{T+1}$ is the corresponding joint plausibility measure and can be obtained as

$$\overline{P}(A) = \sum_{\mathcal{J}} \mathcal{I}(\mathbf{\Psi}(i_1, ..., i_T, j) \cap A) m \left(\mathbf{\Psi}(i_1, ..., i_T, j)\right),$$

Here $\mathcal{I}(\cdot)$ is the indicator function of a set taking the value 1 if its argument is not the empty set and

$$m \left(\mathbf{\Psi}(i_1, ..., i_T, j)\right) = m \left(\mathbf{V}_{1i_1}\right) \cdots m \left(\mathbf{V}_{Ti_T}\right) m \left(\mathbf{r}_j\right). \tag{6}$$

The lower probability $\underline{P}(A)$ is the joint belief measure and can be obtained as[5]

$$\underline{P}(A) = \sum_{\mathcal{J}} \left(1 - \mathcal{I}(\mathbf{\Psi}(i_1, ..., i_T, j) \cap A^c)\right) m \left(\mathbf{\Psi}(i_1, ..., i_T, j)\right).$$

Note that $\mathbf{\Psi}(i_1, ..., i_T, j)$ corresponds to the interval of the NPV denoted by

$$\mathbf{NPV}(i_1, ..., i_T, j) = [\mathrm{NPV}^{\mathrm{L}}(i_1, ..., i_T, j), \mathrm{NPV}^{\mathrm{U}}(i_1, ..., i_T, j)]$$

and obtained from (1) by substituting the intervals of the cash flows with numbers $i_1, ..., i_T$ and the discounted rate with the number j into (1). At that, the computation of the NPV is carried out by means of rules for computing the function of interval-valued variables, i.e.,

$$\mathrm{NPV}^{\mathrm{L}}(i_1, ..., i_T, j) = \inf \left(V_0 + \sum_{k=1}^{T} V_k (1 + r)^{-k} \right), \tag{7}$$

subject to $V_k \in \mathbf{V}_{ki_k}$, $k = 1, ..., T$, and $r \in \mathbf{r}_j$.

The upper bound $\mathrm{NPV}^{\mathrm{U}}$ is computed in the same way by replacing inf on sup. Then if $A = (-\infty, z) \subset \mathbb{R}$, then the lower and upper probabilities that the NPV as the random variable less than z (points of the lower and upper distributions of NPV) are defined as follows:

$$\overline{P}_R(\mathrm{NPV} \leq z) = \sum_{\mathcal{J}} I(\mathrm{NPV}^{\mathrm{L}}(i_1, ..., i_T, j) \leq z) m \left(\mathbf{NPV}(i_1, ..., i_T, j)\right), \tag{8}$$

$$\underline{P}_R(\mathrm{NPV} \leq z) = \sum_{\mathcal{J}} I(\mathrm{NPV}^{\mathrm{U}}(i_1, ..., i_T, j) \leq z) m \left(\mathbf{NPV}(i_1, ..., i_T, j)\right). \tag{9}$$

[5]It is supposed here that $\mathbf{\Psi}(i_1, ..., i_T, j) \cap A^c$ is empty if $\mathbf{\Psi}(i_1, ..., i_T, j) \subset A$.

Here $I(a \leq b) = 1$ if the inequality $a \leq b$ is valid and $I(a \leq b) = 0$ if $a \geq b$. So, by computing the lower and upper bounds for the NPV from $n_1 \cdots n_T m$ possible combinations of intervals of the cash flows and the discounted rate, we can find the lower and upper bounds for $P(\text{NPV} \leq z)$ by arbitrary values of $z \in \mathbb{R}$.

In particular, bounds for the probability of the event NPV ≥ 0 are computed through equalities

$$\overline{P}_R(\text{NPV} \geq 0) = 1 - \underline{P}_R(\text{NPV} \leq 0),\ \underline{P}_R(\text{NPV} \geq 0) = 1 - \overline{P}_R(\text{NPV} \leq 0) \tag{10}$$

due to properties of lower and upper probabilities [16].

By having the lower and upper distributions, the lower $\underline{\mathbb{E}}_R$NPV and upper $\overline{\mathbb{E}}_R$NPV expectations of the NPV can be computed. Note that the distributions of the NPV are step functions such that the lower distribution has jumps at points $z = \text{NPV}^{\text{U}}(i_1, ..., i_T, j)$ and the upper distribution has jumps at points $z = \text{NPV}^{\text{L}}(i_1, ..., i_T, j)$. This implies that

$$\begin{aligned}\overline{\mathbb{E}}_R\text{NPV} &= \int_{\mathbb{R}} z \mathrm{d}\underline{P}_R(\text{NPV} \leq z) \\ &= \sum_{\mathcal{J}} \text{NPV}^{\text{U}}(i_1, ..., i_T, j) m\left(\mathbf{NPV}(i_1, ..., i_T, j)\right),\end{aligned} \tag{11}$$

$$\begin{aligned}\underline{\mathbb{E}}_R\text{NPV} &= \int_{\mathbb{R}} z \mathrm{d}\overline{P}_R(\text{NPV} \leq z) \\ &= \sum_{\mathcal{J}} \text{NPV}^{\text{L}}(i_1, ..., i_T, j) m\left(\mathbf{NPV}(i_1, ..., i_T, j)\right),\end{aligned} \tag{12}$$

subject to $V_k \in \mathbf{V}_{ki_k}$, $k = 1, ..., T$, and $r \in \mathbf{r}_j$.

It is interesting to note that expressions obtained for lower and upper expectations are nothing else but expected utilities based on the Choquet integral [17, 18]. The following solution of the above optimization problems depends on the monotonicity of the NPV as the function of cash flows and the discounted rate.

4.2 Unknown interaction

It is difficult to say about independence of the NPV parameters sometimes when the same experts provide their judgments about the cash flows at different years. At the same time, it is difficult to evaluate the degree of the possible dependence (if it takes place) in this case. When our knowledge about random parameters consists entirely of our knowledge about each of the parameters separately, and we do not know anything about how these parameters are related, unknown interaction is the most preferable method to be on the safe side in analysis [14, 15]. In this case, we have information only about marginal distributions and the upper and lower probabilities of the event NPV $\leq z$ are determined as follows

$$\overline{P}_U(\text{NPV} \leq z) = \max \sum_{\mathcal{J}} I(\text{NPV}^{\text{L}}(i_1, ..., i_T, j) \leq z) m\left(\mathbf{NPV}(i_1, ..., i_T, j)\right), \tag{13}$$

$$\underline{P}_U(\text{NPV} \leq z) = \min \sum_{\mathcal{J}} I(\text{NPV}^{\text{U}}(i_1, ..., i_T, j) \leq z) m\left(\mathbf{NPV}(i_1, ..., i_T, j)\right), \tag{14}$$

subject to

$$m\left(\mathbf{V}_{ki_k}\right) = \sum_{\mathcal{J}(k)} m\left(\mathbf{NPV}(i_1, ..., i_T, j)\right),\ k = 1, ..., T, \tag{15}$$

$$m\left(\mathbf{r}_j\right) = \sum_{\mathcal{J}(j)} m\left(\mathbf{NPV}(i_1, ..., i_T, j)\right). \tag{16}$$

Here $m\left(\mathbf{NPV}(i_1, ..., i_T, j)\right)$ are optimization variables; $\text{NPV}^{\text{L}}(\cdot)$ and $\text{NPV}^{\text{U}}(\cdot)$ are computed in accordance with (7); $\mathcal{J}(k)$ is the set of all index vectors $(i_1, ..., i_T, j)$ with the fixed k-th element i_k; $\mathcal{J}(j)$ is the set of all index vectors $(i_1, ..., i_T, j)$ with the fixed last element j. The constraints to the optimization problems mean that we know only marginal BPAs of subsets, but we can not represent the joint BPAs as the product of marginal ones.

It can be seen from problems (13)-(16) that they are linear and can be solved by the well-known standard simplex method.

4.3 Strong independence

Strong independence is an appropriate choice [14, 15] if outcomes of uncertain parameters are always are stochastically independent and only products of distributions are taken from the set of all joint distributions of variables. This is a stronger generalization of stochastic independence. Strong independence implies that learning the outcome of one experiment does not change our uncertainty about the other experiment, in accordance with the intuitive notion.

Fetz and Oberguggenberger [15] proved a very important property of joint probability measures, which in a simple way and in terms of the NPV computation problem can be formulated as follows. By using some notations introduced for the condition of random set independence, the upper probability of an event $A \subseteq \mathbb{R}^{T+1}$ under the condition of strong independence can be obtained by solving the non-linear optimization problem

$$\overline{P}(A) = \sum_{\mathcal{J}} m\left(\mathbf{\Psi}(i_1, ..., i_T, j)\right) I_A(v_{1i_1}, ..., v_{Ti_T}, r_j) \to \max \tag{17}$$

subject to

$$v_{ki_k} \in \mathbf{V}_{ki_k},\ k = 1, ..., T,\ r_j \in \mathbf{r}_j.$$

Here $I_A(v_{1i_1}, ..., v_{Ti_T}, r_j)$ is the indicator function of the set A. The lower probability $\underline{P}(A)$ is obtained by minimization. If $A = (-\infty, z) \subset \mathbb{R}$, then the lower and upper probabilities that the NPV as the random variable less than z (points of the lower and upper distributions of the NPV) are defined as follows:

$$\underline{P}_S(\text{NPV} \leq z) = \inf \sum_{\mathcal{J}} m\left(\mathbf{NPV}(i_1, ..., i_T, j)\right) I(\text{NPV}(i_1, ..., i_T, j) \leq z), \tag{18}$$

$$\overline{P}_S(\text{NPV} \leq z) = \sup \sum_{\mathcal{J}} m\left(\mathbf{NPV}(i_1, ..., i_T, j)\right) I(\text{NPV}(i_1, ..., i_T, j) \leq z), \tag{19}$$

subject to

$$v_{ki_k} \in \mathbf{V}_{ki_k},\ k = 1, ..., T,\ r_j \in \mathbf{r}_j,$$

$$\text{NPV}(i_1, ..., i_T, j) = V_0 + \sum_{k=1}^{T} \frac{v_{ki_k}}{(1 + r_j)^k}.$$

Here $\text{NPV}(i_1, ..., i_T, j) \in \mathbf{NPV}(i_1, ..., i_T, j)$ is a point in the interval $\mathbf{NPV}(i_1, ..., i_T, j)$ obtained by substituting the values v_{ki_k} and r_j into (1). Generally, problems (18) and (19) are non-linear and their solution is a difficult task. In contrast to random set independence, every optimization variable v_{ki_k} takes place in $n_1 \cdots n_T m / n_k$ terms of the sum in the objective functions. However, the monotonicity properties of the NPV allow us to significantly simplify the problems.

4.4 Monotone NPV, random set independence and unknown interaction

It is obvious that the monotonicity of the NPV significantly simplifies the analysis of its expectation and probability $\Pr(\text{NPV} \geq 0)$. Therefore, the case of a monotone NPV is analyzed in the first place.

If the NPV is monotone, then the bounds $\text{NPV}^{\text{L}}(i_1, ..., i_T, j)$ and $\text{NPV}^{\text{U}}(i_1, ..., i_T, j)$ can be easily computed as follows:

$$\text{NPV}^{\text{L}}(i_1, ..., i_T, j) = V_0 + \sum_{k=1}^{T} V_{ki_k}^{\text{L}} (1 + r_j^{\text{U}})^{-k}, \tag{20}$$

$$\text{NPV}^{\text{U}}(i_1, ..., i_T, j) = V_0 + \sum_{k=1}^{T} V_{ki_k}^{\text{U}} (1 + r_j^{\text{L}})^{-k}. \tag{21}$$

By substituting these bounds into (8)-(9) and (11)-(12), by using (6) and by looking over all $n_1 \cdots n_T m$ elements of the set $\mathcal{J}$, we obtain the bounds for the probability $P(\text{NPV} \geq 0)$ and expectation $\mathbb{E}\text{NPV}$.

Example 2 *Suppose that the following information about project is available:*

$$V_0 = -2,\ \mathbf{V}_{11} = [0, 3],\ c_{11} = 4,\ \mathbf{V}_{12} = [1.5, 2],\ c_{12} = 1,\ n_1 = 2,\ N_1 = 5,$$

$$\mathbf{V}_{21} = [3, 4],\ c_{21} = 2,\ n_2 = 1,\ N_2 = 2,$$

$$\mathbf{r}_1 = [0.1, 0.2],\ d_1 = 1,\ \mathbf{r}_2 = [0.15, 0.25],\ d_2 = 1,\ m = 2,\ M = 2.$$

Then the NPV is monotone by all possible values of V_1 and V_2 because the cash flows are non-negative. The set $\mathcal{J}$ consists of the following elements: $(1,1,1)$, $(2,1,1)$, $(1,1,2)$, $(2,1,2)$. The BPAs of intervals are

$$m(\mathbf{V}_{11}) = 4/5,\ m(\mathbf{V}_{12}) = 1/5,\ m(\mathbf{V}_{21}) = 1,\ m(\mathbf{r}_1) = 1/2,\ m(\mathbf{r}_2) = 1/2.$$

Then

$$\mathrm{NPV}^{\mathrm{L}}(1,1,1) = -2 + 0 \cdot (1+0.2)^{-1} + 3 \cdot (1+0.2)^{-2} = 0.08,$$
$$\mathrm{NPV}^{\mathrm{U}}(1,1,1) = -2 + 3 \cdot (1+0.1)^{-1} + 4 \cdot (1+0.1)^{-2} = 4.03,$$
$$m(\mathbf{NPV}(1,1,1)) = 2/3 \cdot 1 \cdot 1/2 = 1/3.$$

Similarly, we get $\mathrm{NPV}^{\mathrm{L}}(\cdot)$ *and* $\mathrm{NPV}^{\mathrm{U}}(\cdot)$ *for all possible vectors (i_1, i_2, j):* $\mathrm{NPV}^{\mathrm{L}}(2,1,1) = 1.33$, $\mathrm{NPV}^{\mathrm{L}}(2,1,2) = 1.12$, $\mathrm{NPV}^{\mathrm{L}}(1,1,2) = -0.08$, $\mathrm{NPV}^{\mathrm{U}}(2,1,1) = 3.12$, $\mathrm{NPV}^{\mathrm{U}}(2,1,2) = 2.76$, $\mathrm{NPV}^{\mathrm{U}}(1,1,2) = 3.63$. *It follows from (8)-(9) that* $\underline{P}_R(\mathrm{NPV} \leq 0) = 0$ *and*

$$\overline{P}_R(\mathrm{NPV} \leq 0) = m(\mathbf{NPV}(1,1,2)) = 0.4.$$

Hence $\underline{P}_R(\mathrm{NPV} \geq 0) = 0.6$ *and* $\overline{P}_R(\mathrm{NPV} \geq 0) = 1$. *The lower and upper expectations can be found from (11)-(12) and they are* $\underline{\mathbb{E}}_R\mathrm{NPV} = 0.26$, $\overline{\mathbb{E}}_R\mathrm{NPV} = 3.66$.

In the case of unknown interaction, by substituting (20)-(21) into objective functions (13)-(14), we get the standard linear programming problems.

Example 3 *Let us return to Example 2 and find the characteristics of NPV under condition of unknown interaction. If we denote $x_{ijk} = m(\mathbf{NPV}(i,j,k))$ for short, then the linear programming problem for computing the upper probability is*

$$\overline{P}_U(\mathrm{NPV} \leq 0) = x_{112} \to \max,$$

subject to $x_{ijk} \geq 0$, $i = 1,2$, $j = 1$, $k = 1,2$, and

$$\begin{aligned} 4/5 &= x_{111} + x_{112}, \\ 1/5 &= x_{211} + x_{212}, \\ 1 &= x_{111} + x_{112} + x_{211} + x_{212}, \\ 1/2 &= x_{111} + x_{211}, \\ 1/2 &= x_{112} + x_{212}. \end{aligned}$$

Hence $\overline{P}_U(\mathrm{NPV} \leq 0) = 0.5$. *The lower probability is* 0 *because* $\mathrm{NPV}^{\mathrm{U}}(\cdot) \geq 0$ *for all possible i, j, k. This implies that* $\underline{P}_U(\mathrm{NPV} \geq 0) = 0.5$ *and* $\overline{P}_U(\mathrm{NPV} \geq 0) = 1$, *and*

$$[\underline{P}_R(\mathrm{NPV} \geq 0), \overline{P}_R(\mathrm{NPV} \geq 0)] \subseteq [\underline{P}_U(\mathrm{NPV} \geq 0), \overline{P}_U(\mathrm{NPV} \geq 0)].$$

4.5 Monotone NPV, strong independence

Note that the indicator functions in problem (19) are non-increasing with z. Moreover, every coefficient $m\left(\mathbf{NPV}(i_1, ..., i_T, j)\right)$ is positive. Therefore, if NPV is monotone, then by taking $v_{ki_k} = V^{\mathrm{U}}_{ki_k}$ for all $k = 1, ..., T$, and $r_j = r^{\mathrm{L}}_j$ as optimal values, we get maximal values of all possible indicator functions $I(\mathrm{NPV}(i_1, ..., i_T, j) \leq z)$. This implies that the upper probability $\overline{P}_S(\mathrm{NPV} \leq z)$ coincides with (8). The same can be said about the lower probability $\underline{P}_S(\mathrm{NPV} \leq z)$. Then the lower $\underline{\mathbb{E}}_S\mathrm{NPV}$ and upper $\overline{\mathbb{E}}_S\mathrm{NPV}$ expectations are determined by (11)-(12). The considered bounds in the case of strong independence do not differ from the same bounds in the case of random set independence. This feature is valid only for the monotone NPV. We do not consider an example here because it coincides with Example 2.

4.6 Non-monotone NPV, random set independence and unknown interaction

If NPV is non-monotone, we can not find $\mathrm{NPV}^{\mathrm{L}}(i_1,...,i_T,j)$ and $\mathrm{NPV}^{\mathrm{U}}(i_1,...,i_T,j)$ by substituting the corresponding bounds of intervals $\mathbf{V}_{1i_1},...,\mathbf{V}_{Ti_T},\mathbf{r}_j$ into (1). This implies that optimization problem (7) has to be solved for every element of $\mathcal{J}$ for computing $\mathrm{NPV}^{\mathrm{L}}(\cdot)$ and the similar problem has to be solved for computing $\mathrm{NPV}^{\mathrm{U}}(\cdot)$. However, NPV is always non-decreasing with the cash flows. This implies that the optimization problems can be simplified

$$\mathrm{NPV}^{\mathrm{L}}(i_1,...,i_T,j) = \inf_{r\in\mathbf{r}_j}\left(V_0 + \sum_{k=1}^{T} V_{ki_k}^{\mathrm{L}}(1+r)^{-k}\right),$$

$$\mathrm{NPV}^{\mathrm{U}}(i_1,...,i_T,j) = \sup_{r\in\mathbf{r}_j}\left(V_0 + \sum_{k=1}^{T} V_{ki_k}^{\mathrm{U}}(1+r)^{-k}\right).$$

Hence, the functions of one variable are analyzed for computing their maximum and minimum in the interval $\mathbf{r}_j$. At that, expressions (8)-(9) and (11)-(12) remain without changes.

The same calculations of $\mathrm{NPV}^{\mathrm{L}}(\cdot)$ and $\mathrm{NPV}^{\mathrm{U}}(\cdot)$ are carried out in the case of unknown interaction.

Example 4 *Let us return to Example 1 and analyze NPV under condition of random set independence. Here*

$$m(\mathbf{V}_{11}) = 3/4,\ m(\mathbf{V}_{12}) = 1/4,\ m(\mathbf{V}_{21}) = 1,\ m(\mathbf{r}_1) = 1/2,\ m(\mathbf{r}_2) = 1/2.$$

It can be seen that NPV is non-monotone when, for instance, $V_1 = -4$ and $V_2 = 2.4$. The function has the minimum at point $r = 0.2$. Then

$$\mathrm{NPV}^{\mathrm{L}}(1,1,1) = \inf_{r\in[0.1,0.2]}\left(-4(1+r)^{-1} + 2.4\cdot(1+r)^{-2}\right) = -1.67,$$

$$\mathrm{NPV}^{\mathrm{L}}(1,1,2) = \inf_{r\in[0.15,0.25]}\left(-4(1+r)^{-1} + 2.4\cdot(1+r)^{-2}\right) = -1.67,$$

Similarly, we get $\mathrm{NPV}^{\mathrm{L}}(2,1,2) = -0.024$, $\mathrm{NPV}^{\mathrm{L}}(2,1,1) = 0.042$, $\mathrm{NPV}^{\mathrm{U}}(1,1,1) = 3.06$, $\mathrm{NPV}^{\mathrm{U}}(1,1,2) = 2.68$, $\mathrm{NPV}^{\mathrm{U}}(2,1,1) = 6.69$, $\mathrm{NPV}^{\mathrm{U}}(2,1,2) = 6.16$. *It follows from (8)-(9) that* $\underline{P}_R(\mathrm{NPV}\leq 0) = 0$ *and*

$$\overline{P}_R(\mathrm{NPV}\leq 0) = m(\mathbf{NPV}(1,1,1)) + m(\mathbf{NPV}(1,1,2)) + m(\mathbf{NPV}(2,1,2)) = 0.875.$$

Hence $\underline{P}_R(\mathrm{NPV}\geq 0) = 0.125$ *and* $\overline{P}_R(\mathrm{NPV}\geq 0) = 1$. *The lower and upper expectations can be found from (11)-(12) and they are* $\underline{\mathbb{E}}_R\mathrm{NPV} = -1.24$, $\overline{\mathbb{E}}_R\mathrm{NPV} = 3.77$.

Example 5 *Let us return to Example 1 and find the characteristics of NPV under condition of unknown interaction by using some intermediate results of Example 4. If we denote $x_{ijk} = m(\mathbf{NPV}(i,j,k))$, then the linear programming problem for computing the upper probability is*

$$\overline{P}_U(\mathrm{NPV}\leq 0) = x_{112} + x_{112} + x_{212} \to \max,$$

subject to $x_{ijk}\geq 0$, $i=1,2$, $j=1$, $k=1,2$, and

$$\begin{aligned}
3/4 &= x_{111} + x_{112},\\
1/4 &= x_{211} + x_{212},\\
1 &= x_{111} + x_{112} + x_{211} + x_{212},\\
1/2 &= x_{111} + x_{211},\\
1/2 &= x_{112} + x_{212}.
\end{aligned}$$

Hence $\overline{P}_U(\mathrm{NPV}\leq 0) = 1$. *The lower probability is* 0 *because* $\mathrm{NPV}^{\mathrm{U}}(\cdot)\geq 0$ *for all possible i,j,k. This implies that* $\underline{P}_U(\mathrm{NPV}\geq 0) = 0$ *and* $\overline{P}_U(\mathrm{NPV}\geq 0) = 1$, *i.e., we can not make any decision about the project under available data in the case of unknown interaction.*

4.7 Non-monotone NPV, strong independence

For solving optimization problems (18)-(19) in the case of strong independence, we also use the properties that NPV is non-decreasing with the cash flows and the indicator functions in (18)-(19) are non-increasing with z. Then

$$\underline{P}_S(\text{NPV} \leq z) = \inf \sum_{\mathcal{J}} m\left(\mathbf{NPV}(i_1, ..., i_T, j)\right) I(\text{NPV}^{\text{U}}(i_1, ..., i_T, j) \leq z), \tag{22}$$

$$\overline{P}_S(\text{NPV} \leq z) = \sup \sum_{\mathcal{J}} m\left(\mathbf{NPV}(i_1, ..., i_T, j)\right) I(\text{NPV}^{\text{L}}(i_1, ..., i_T, j) \leq z), \tag{23}$$

subject to $r_j \in \mathbf{r}_j$, $j = 1, ..., m$.

Here $\text{NPV}^{\text{L}}(\cdot)$ and $\text{NPV}^{\text{U}}(\cdot)$ are computed by substituting $V_{ki_k}^{\text{L}}$ and $V_{ki_k}^{\text{U}}$ into (1), respectively.

Denote $\mathcal{J}_0 = \{(i_1, ..., i_T) : \ i_k = 1, ..., n_k, \ k = 1, ..., T\}$. Objective functions (22)-(23) can be rewritten as

$$\underline{P}_S(\text{NPV} \leq z) = \sum_{j=1}^{m} m\left(\mathbf{r}_j\right) \inf_{r_j \in \mathbf{r}_j} \sum_{\mathcal{J}_0} \prod_{k=1}^{T} m\left(\mathbf{V}_{ki_k}\right) I(\text{NPV}^{\text{U}}(i_1, ..., i_T, j) \leq z), \tag{24}$$

$$\overline{P}_S(\text{NPV} \leq z) = \sum_{j=1}^{m} m\left(\mathbf{r}_j\right) \sup_{r_j \in \mathbf{r}_j} \sum_{\mathcal{J}_0} \prod_{k=1}^{T} m\left(\mathbf{V}_{ki_k}\right) I(\text{NPV}^{\text{L}}(i_1, ..., i_T, j) \leq z). \tag{25}$$

We get m optimization problems for computing every bound and every optimization problem has one variable r_j restricted by the interval $\mathbf{r}_j$. It can be seen from (24)-(25) that the main difference between optimization problems in the cases of strong independence and random set independence is the following. By assuming random set independence, we search the minimum or maximum of NPV for every element $(i_1, ..., i_T, j)$ of $\mathcal{J}$ whereas, by assuming strong independence, we search the optimal values for every j. The above implies that the feasible region is smaller by strong independence and, therefore, strong independence leads to more precise results.

Example 6 *Let us return to Example 1 and analyze NPV under condition of strong independence. The set* $\mathcal{J}_0$ *consists of the following elements:* $(1,1)$, $(2,1)$. *Let us consider the first optimization problem* ($j = 1$) *for computing* $\overline{P}_S(\text{NPV} \leq z)$

$$\sup_{r_1 \in \mathbf{r}_1} \sum_{\mathcal{J}_0} m\left(\mathbf{V}_{1i_1}\right) m\left(\mathbf{V}_{2i_2}\right) I(\text{NPV}^{\text{L}}(i_1, i_2, 1) \leq z).$$

The optimization problem can be rewritten as

$$\frac{3}{4} \cdot 1 \cdot I(\text{NPV}^{\text{L}}(1,1,1) \leq z) + \frac{1}{4} \cdot 1 \cdot I(\text{NPV}^{\text{L}}(2,1,1) \leq z) \to \max,$$

subject to $r \in [0.1, 0.2]$.

Here

$$\text{NPV}^{\text{L}}(1,1,1) = \left(-4(1+r)^{-1} + 2.4 \cdot (1+r)^{-2}\right),$$

$$\text{NPV}^{\text{L}}(2,1,1) = \left(-1.95(1+r)^{-1} + 2.4 \cdot (1+r)^{-2}\right).$$

By taking $z = 0$ *and* $r \in [0.1, 0.2]$, *we get the optimal value of the objective function* $3/4$. *If* $r \in [0.15, 0.25]$, *i.e.,* $r \in \mathbf{r}_2$ *then the optimal value of the same objective function is* 1. *Hence*

$$\overline{P}_S(\text{NPV} \leq z) = 3/4 \cdot 1/2 + 1 \cdot 1/2 = 0.875.$$

The lower probability is $\underline{P}_S(\text{NPV} \leq 0) = 0$. *This implies that* $\underline{P}_S(\text{NPV} \geq 0) = 0.125$ *and* $\overline{P}_S(\text{NPV} \geq 0) = 1$.

It is interesting to note that the discount rate is precise in many applications, especially, if one chooses an optimal project among the possible ones with a given discount rate. It has been pointed out that the NPV is a monotone function in this case and its calculation does not meet difficulties. Moreover, we obtain the same results under conditions of strong independence and random set independence.

5 Applying information about experts

The above methods have been developed under condition that the experts provided interval-valued estimates about the parameters of NPV are unknown and possible information about experts is not used. As a result, the BPAs of estimates are elicited as empirical BPAs based on relative frequencies c_{ki} (d_j) of the intervals $\mathbf{V}_{ki}$ ($\mathbf{r}_j$) (see (2)). However, the additional information about experts in the form of their *weights* might be very useful and would allow us to make more accurate results of the NPV analysis[6].

Suppose that Q experts provide information about every parameter of NPV such that the q-th expert supplies at most one interval from $\mathcal{V}_k$, $k = 1, ..., T$, or from $\mathcal{R}$ (see (4) and (5)). Suppose that the interval $\mathbf{V}_{ki}$ is given by the experts with numbers belonging to the set W_{ki}. At that, there holds $W_{ki} \cap W_{kj} = \varnothing$ for $i \neq j$. Since an expert supplies at most one interval for a parameter of NPV, then every set W_{ki} can not contain two or more identical numbers. In particular, an expert with number q may not supply estimates for some of the NPV parameters and W_{ki} does not contain q in this case. The same can be said about discount rate: the interval $\mathbf{r}_j$ is provided by experts with numbers from W_j.

The quality of the q-th expert is measured by weight w_q such that there holds $\sum_{q=1}^{Q} w_q = 1$. In order to take into account the information about experts, we return to the multivalued mapping explanation of belief functions. Let us consider in detail the universal set Ω. Every interval corresponds to a point ω from this set. At that, the probability of ω is defined by the number of experts c_{ki} (d_j) provided the interval. If we consider all c_{ki} coinciding points separately, then every point has the probability $1/N_k$. This corresponds to the case when all experts provided the corresponding interval $\mathbf{V}_{ki}$ are identical. However, if to suppose that experts have weights, then we can attribute to every point the corresponding weight. By uniting c_{ki} identical points corresponding to the interval $\mathbf{V}_{ki}$, we attribute the total weight

$$\sum_{q=1}^{Q} w_q I(q \in W_{ki}).$$

Here $I(q \in W_{ki})$ is the indicator function taking the value 1 if $q \in W_{ki}$. Hence, the BPA of the interval $\mathbf{V}_{ki}$ can be updated as follows:

$$m(\mathbf{V}_{ki}) = \frac{c_{ki}}{N_k} \sum_{q=1}^{Q} w_q I(q \in W_{ki}).$$

Since $I(q \in W_{ki})$ may be 0 for some q and the condition $\sum_{i=1}^{n_k} m(\mathbf{V}_{ki}) = 1$ has to be satisfied, we normalize the BPAs as follows:

$$m(\mathbf{V}_{ki}) = \frac{c_{ki}}{C_k N_k} \sum_{q=1}^{Q} w_q I(q \in W_{ki}),$$

where

$$C_k = \sum_{i=1}^{n_k} \sum_{q=1}^{Q} w_q I(q \in W_{ki}).$$

The above normalization is required if $Q > N_k$. The BPAs for the discount rate have the same form

$$m(\mathbf{r}_j) = \frac{d_j}{CM} \sum_{q=1}^{Q} w_q I(q \in W_j),\ C = \sum_{j=1}^{m} \sum_{q=1}^{Q} w_q I(q \in W_j).$$

It should be noted that the accounting of the expert quality changes BPAs of the provided intervals and does not impact on other calculations of NPV.

[6]There exist a lot of methods for taking into account the quality of experts[19] when the expert estimates are point-valued. We apply one of the most popular method based on weighted averaging to interval-valued estimates.

6 Cautious analysis of NPV by using the imprecise Dirichlet model

Definition (2) of BPAs can be used when the number of estimates or observations of the cash flows and the discount rate is rather large. However, this condition may be violated because it is usually difficult to find a lot of experts estimating a specific project. If the number of estimates is small, inferences become too incautious. Let us add one judgment $[8,10]$ of the cash flow V_2 to available data in Example 1. Then we get $\underline{P}_R(\mathrm{NPV} \geq 0) = 0.563$ and $\overline{P}_R(\mathrm{NPV} \geq 0) = 1$ under condition of random set independence. The same bounds are obtained by strong independence. One can see that the lower bound significantly differs from the lower bound $\underline{P}_R(\mathrm{NPV} \geq 0) = 0.125$ given in Example 4. In other words, one judgment might cardinally change the decision situation. On the other hand, if we return to initial data in Example 1, but add many intervals $[2.4, 7]$ of the cash flow V_2. In this case, we get the bounds $\underline{P}_R(\mathrm{NPV} \geq 0) = 0.125$ and $\overline{P}_R(\mathrm{NPV} \geq 0) = 1$, which do not differ from those of obtained in Example 4. This implies that the number of intervals $[2.4, 7]$ does not impact on the final results under considered conditions.

In order to overcome these difficulties, we shall apply the imprecise Dirichlet model [10], extending belief and plausibility functions such that a lack of sufficient data can be taken into account [20, 21].

Let us return to the multivalued mapping with the universal set $\Omega = \{\omega_1, ..., \omega_n\}$. The probability measure $P(\omega)$ on the universal set Ω has been defined as $P(\omega_i) = c_i/N$. Now we suppose that the probability of every point ω_i is a random variable governed by the Dirichlet distribution. Moreover, due to the lack of precise information about parameters of the Dirichlet distribution, the random variables are governed by a set of the Dirichlet distributions called the imprecise Dirichlet model.

Let $\Omega = \{\omega_1, ..., \omega_n\}$ be a set of possible outcomes ω_j. Assume the *standard multinomial model*: N observations are independently chosen from Ω with an identical probability distribution $\Pr\{\omega_j\} = \theta_j$ for $j = 1, ..., n$, where each $\theta_j \geq 0$ and $\sum_{j=1}^n \theta_j = 1$. Denote $\theta = (\theta_1, ..., \theta_n)$. Let c_j denote the number of observations of ω_j in the N trials, so that $c_j \geq 0$ and $\sum_{j=1}^n c_j = N$.

The *Dirichlet* $(s, \mathbf{t})$ *prior distribution* for θ, where $\mathbf{t} = (t_1, ..., t_n)$, has probability density function [22]

$$p(\theta) = \Gamma(s) \left(\prod_{j=1}^n \Gamma(st_j) \right)^{-1} \cdot \prod_{j=1}^n \theta_j^{st_j - 1}.$$

Here the parameter $t_i \in (0, 1)$ is the mean of θ_i under the Dirichlet prior; the hyperparameter $s > 0$ determines the influence of the prior distribution on posterior probabilities; the vector $\mathbf{t}$ belongs to the interior of the n-dimensional unit simplex denoted by $S(1, n)$; $\Gamma(\cdot)$ is the Gamma-function.

Walley [10] pointed out several reasons for using a set of Dirichlet distributions to model prior ignorance about probabilities θ:

1. Dirichlet prior distributions are mathematically tractable because they generate Dirichlet posterior distributions;

2. sets of Dirichlet distributions are very rich, because they produce the same inferences as their convex hulls and any prior distribution can be approximated by a finite mixture of Dirichlet distributions;

3. the most common Bayesian models for prior ignorance about probabilities θ are Dirichlet distributions.

The *imprecise Dirichlet model* (IDM) is defined by Walley [10] as the set of all Dirichlet $(s, \mathbf{t})$ distributions such that $\mathbf{t} \in S(1, n)$. We will see below that the choice of this model allows us to model the fact that we do not know the a priori probabilities of events. For the IDM, the *hyperparameter* s determines how quickly upper and lower probabilities of events converge as statistical data accumulate. Walley [10] defined s as a number of observations needed to reduce the imprecision (difference between upper and lower probabilities) to half its initial value. Smaller values of s produce faster convergence and stronger conclusions, whereas large values of s produce more cautious inferences. At the same time, the value of s must not depend on n or a number of observations. The detailed discussion concerning the parameter s and the IDM can be found in [23, 10].

Let A be any non-trivial subset of a sample space $\{\omega_1, ..., \omega_n\}$, i.e., A is not empty and $A \neq \Omega$, and let $c(A)$ denote the observed number of occurrences of A in the N trials, $c(A) = \sum_{\omega_j \in A} c_j$. Then the predictive probability $P(A|s)$ under the Dirichlet posterior distribution is[7]

$$P(A|s) = (c(A) + st(A)) \,/\, (N + s)\,.$$

Here $t(A) = \sum_{\omega_j \in A} t_j$. It should be noted that $P(A|s) = 0$ if $A = \varnothing$ and $P(A|s) = 1$ if $A = \Omega$.

By maximizing and minimizing $P(A|s)$ over $\mathbf{t} \in S(1, n)$, we obtain the posterior upper and lower probabilities of A:

$$\underline{P}(A|s) = c(A)/\,(N + s)\,,\ \overline{P}(A|s) = (c(A) + s)/\,(N + s)\,.$$

Before making any observations, $c(A) = N = 0$, so that $\underline{P}(A|s) = 0$ and $\overline{P}(A|s) = 1$ for all non-trivial events A. Therefore, by using the IDM, we do not need to choose one specific prior. In contrast, the objective Bayesian approach [24] aims at modeling prior ignorance about the chances θ by characterizing prior uncertainty by a single prior probability distribution.

Why and when should we use the IDM instead of the "precise" Dirichlet distribution? Suppose that we toss a coin five times and have 3 heads and 2 tails. By using the "precise" Dirichlet distribution (the Dirichlet distribution is none other than the beta-distribution in case of two possibilities $n = 2$), we get $P(heads) = 3/5 \neq 1/2$. At the same time, if we take the IDM with $s = 1$, then $\underline{P}(heads|1) = 3/6$ and $\overline{P}(heads|1) = 4/6$. It can be seen from these results that

$$\underline{P}(heads|1) \leq 1/2 \leq \overline{P}(heads|1).$$

In other words, the IDM provides lower and upper bounds for probabilities of events when the number of observations is rather small and the resulting inference might be too incautious.

By returning to the multivalued mapping and by using the IDM, we get for $A \subset \Omega$

$$\underline{P}(A|s) = (N + s)^{-1} \sum_{\omega_j \in X_*} c_j = \frac{N \cdot Bel(A)}{N + s} = \varkappa \cdot Bel(A).$$

Here $\varkappa = (1 + s/N)^{-1}$ and $\varkappa \in [0, 1]$. If $A = \Omega$, then $\underline{P}(A|s) = 1$. The upper probability of A can be obtained in the same way:

$$\overline{P}(A|s) = \left(\sum_{\omega_j \in X^*} c_j + s \right) /(N + s) = \frac{N \cdot Pl(A) + s}{N + s} = 1 - \varkappa(1 - Pl(A)).$$

It is interesting to point out that $\underline{P}(A|s)$ and $\overline{P}(A|s)$ are belief and plausibility functions with the BPA $m^*(A_i) = c_i/(N + s)$ for every A_i and the additional BPA $m^*(\Omega) = s/(N + s)$, i.e., $\underline{P}(A|s)$ and $\overline{P}(A|s)$ can be obtained as standard belief and plausibility functions under condition that there are s additional observations $A_{n+1} = \Omega$. If we denote $m(A_i) = c_i/N$, then $m^*(A_i) = \varkappa \cdot m(A_i)$, and

$$\underline{P}(A|s) = \sum_{A_i : A_i \subseteq A} m^*(A_i),\ \overline{P}(A|s) = m^*(\Omega) + \sum_{A_i : A_i \cap A \neq \varnothing} m^*(A_i).$$

The above implies that the use of the IDM leads to replacing the BPAs of the interval-valued cash flows provided by experts on BPAs m^* with supplementing the additional estimate of the interval $[\inf V_k, \sup V_k]$. The same concerns the discount rate.

Let us consider a special case when standard belief and plausibility functions may give improper conclusions. Suppose that we have an interval-valued estimate $\mathbf{V}_{k1}$ of the k-th cash flow. The main reason of the above is that the probabilities outside interval-valued estimates in random set theory is 0. According to (2) and (3), the corresponding belief and plausibility functions are $Bel(\mathbf{V}_{k1}) = Pl(\mathbf{V}_{k1}) = 1$. This implies that we

[7]It is necessary to emphasize that the notion $P(A|(n_1, ..., n_n), \mathbf{t}, s)$ for the predictive probability is more correct. However, we use notation $P(A|s)$ for short.

completely believe this one estimate and suppose that a value of the cash flow outside $\mathbf{V}_{k1}$ can not occur: this conclusion may be too risky. This can be avoided by using the IDM ($s > 0$). Indeed, if we take $s = 1$, then $\underline{P}(\mathbf{V}_{k1}|1) = 1/(1+s) = 0.5$ and $\overline{P}(\mathbf{V}_{k1}|1) = 1$. It is worth noting that if we have c_{k1} identical estimates, then the belief and plausibility functions are the same $Bel(\mathbf{V}_{k1}) = Pl(\mathbf{V}_{k1}) = 1$. This implies that the lower and upper probabilities do not depend on the number of expert estimates while $\underline{P}(\mathbf{V}_{k1}|s) = c_{k1}/(c_{k1}+s)$ and $\overline{P}(\mathbf{V}_{k1}|s) = 1$.

It can be seen from the above that the IDM presupposes that there is some non-zero probability for points outside the given intervals. This is a very important property which allows us to overcome some difficulties of standard models of random set theory.

At the same time, if the final aim is to find lower and upper expected values of NPV, then we have to restrict the set of possible values of the cash flows and the discounted rate by some lower and upper bounds $\inf V_k$, $\sup V_k$, $\inf r$, $\sup r$ because unrestricted bounds lead to unrestricted bounds for the expectation of the NPV due to non-zero probabilities of all points outside the given interval-valued estimates, i.e., $\underline{\mathbb{E}}\text{NPV} \to -\infty$, $\overline{\mathbb{E}}\text{NPV} \to \infty$. Moreover, even by restricting the cash flows and the discounted rate, we get the lower and upper expectations strongly depending on $\inf V_k$, $\sup V_k$, $\inf r$, $\sup r$. This shortcoming is not of importance when our aim is lower and upper probabilities that NPV exceeds some certain value.

Example 7 *Let us return to Example 1 and analyze NPV under condition of random set independence by using the IDM. Now by adding one judgment* $[8, 10]$ *of the cash flow* V_2 *to available data in Example 1 and taking* $s = 1$*, we get*

$$\varkappa_1 = 0.8,\ m^*(\mathbf{V}_{11}) = 0.6,\ m^*(\mathbf{V}_{12}) = 0.2,\ m^*(U_1) = 0.2,$$

$$\varkappa_2 = 0.5,\ m^*(\mathbf{V}_{21}) = 0.5,\ m^*(U_2) = 0.5,$$

$$\varkappa_3 = 2/3,\ m^*(\mathbf{r}_1) = 1/3,\ m^*(\mathbf{r}_2) = 1/3,\ m^*(U_3) = 1/3.$$

Here $U_k = [\inf V_k, \sup V_k]$*,* $k = 1, 2$*,* $U_3 = [\inf r, \sup r]$*. If we assume that* $\inf V_k = -10$*,* $\sup V_k = 10$*,* $\inf r = 0$*,* $\sup r = 1$*, then* $\underline{P}_R(\text{NPV} \geq 0|1) = 0.29$ *and* $\overline{P}_R(\text{NPV} \geq 0|1) = 1$*,* $\underline{\mathbb{E}}_R\text{NPV} = -3.86$*,* $\overline{\mathbb{E}}_R\text{NPV} = 4.24$*. It can be seen from the results that the bounds are closed to the bounds obtained in Example 4 without the "extreme" judgment* $[8, 10]$*. This implies that the IDM allows us to soften the possible outliers and contradictory estimates. If we take* $s = 2$*, then* $\underline{P}_R(\text{NPV} \geq 0|2) = 0.18$ *and* $\overline{P}_R(\text{NPV} \geq 0|2) = 1$*. Therefore, it can be concluded that the IDM results more cautious and stable bounds for the characteristics of the NPV.*

7 Concluding remarks

NPV is a rather common measure of the project risk analysis and, therefore, methods for its defining and efficient computing by different initial data about its parameters is an important task. A method for computing the NPV has been studied in the paper under condition that the initial data are interval-valued judgments provided by experts. In conclusion, I would like to stress on the following questions:

1. *Computation.* One can see from the proposed results that the computation of NPV requires to look over all combinations of indices (the set $\mathcal{J}$). This is one of the most hard places from computational point of view. However, the number of expert judgments in real applications is usually small enough and the computation does not take much time. Moreover, there exist a lot of efficient standard recursive procedures which can simply be realized for looking over all elements of $\mathcal{J}$.

2. *Interval-valued NPV characteristics.* It can be seen from the results that the obtained probabilities or expectations are interval-valued. Moreover, the resultant intervals may be rather wide. How to make decision in this case? First of all, the possible large imprecision of results reflects the imprecision of initial data or their contradiction when experts provide contradictory judgments. There are a number of methods for processing interval-valued results and most methods are based on computing some exact value of an interval. One of the most attractive and justified methods is a method using the so-called caution parameter [25] or the parameter of pessimism η which has the same meaning as the optimism parameter in Hurwicz criterion. For example, if we obtain the lower $\underline{P}(\text{NPV} \geq 0)$ and upper $\overline{P}(\text{NPV} \geq 0)$ probabilities, then the exact value of the probability can be computed as $\eta\underline{P}(\text{NPV} \geq 0) + (1 -$

$\eta)\overline{P}(\text{NPV} \geq 0)$. When several projects are compared, then various criteria of decision making with imprecise probabilities might be used, for instance, Walley's maximality rule, E-admissibility, etc. [26].

3. *Independence.* Three types of independence have been considered in the paper. In accordance with this, the obvious question arises what type of independence should be taken into account. It is difficult to unambiguously answer. On one hand, it is possible to say that unknown interaction is the most conservative type of independence, but it results the most imprecise intervals that may lead to impossibility to make decision. On the other hand, strong independence is the most cautious type, but its use reduces the imprecision of results. In other words, unknown interaction and strong independence are two extreme types of independence and random set independence can be regarded as a neutral type.

4. *Point-valued data.* Sometimes, experts supply point-valued data in place of interval-valued ones. In this case, the proposed approach for computing the NPV can be applied to this type of data under condition that a point is a special (degenerate) case of an interval. It is obvious that the computation procedures for computing the lower and upper probabilities or expectations become simpler because it is not necessary to search minimal (maximal) values of the NPV in intervals corresponding to every vector from the set $\mathcal{J}$ (except the case of using the IDM). If the IDM is not used, then the resultant probabilities are precise under conditions of random set independence or strong independence. However, this can be viewed as a shortcoming because, by accepting the precise results, we completely believe in estimates provided by overconfident [27] experts. In order to avoid this shortcoming, the IDM should be used. This model adds intervals $[\inf V_k, \sup V_k]$, $[\inf r, \sup r]$ as the estimates of overcautious [27] experts. These additional imprecise estimates "compensate" precise data and allow us to make a more prudent decision. Suppose that we have point-valued estimates for V_1: $\{-3,-3,-3,1\}$, for V_2: $\{4,3\}$, for r: $\{0.1, 0.2\}$. By assuming $V_0 = 0$ and representing every point-valued estimate as the degenerate interval, we get identical lower and upper bounds for the probability of event NPV ≥ 0 under conditions of random set independence and strong independence

$$\underline{P}_{R,S}(\text{NPV} \geq 0) = \overline{P}_{R,S}(\text{NPV} \geq 0) = 0.625.$$

These results are unbelievable because the initial information is very scarce. However, if we use the IDM with $s = 1$ and $\inf V_k = -10$, $\sup V_k = 10$, $\inf r = 0$, $\sup r = 1$, then

$$\underline{P}_{R,S}(\text{NPV} \geq 0) = 0.27,\ \overline{P}_{R,S}(\text{NPV} \geq 0) = 0.87.$$

The above short numerical example illustrates that the IDM gives more realistic results.

5. *Interval-valued probability distributions of the parameters.* Sometimes, the lower and upper continuous probability distributions of the NPV parameters are known. These distributions can be regarded as a p-box [12]. Ferson *et al* [12] and Kriegler and Held [28] proposed to approximate the lower and upper probability distributions by random sets. Moreover, Kriegler and Held [28] give an efficient algorithm for the approximation. Consequently, we can analyze the case of interval-valued probability distributions of the NPV parameters by means of the approach proposed in the paper. At that, the IDM loses its force because the number of intervals may be rather large and depends on accuracy of the approximation.

Some of the above questions are very shortly considered here and could be regarded as directions for further work.

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