Interval assessments in structural reliability using the imprecise Dirichlet model

Lev V. Utkin[1]* and Vladimir S. Utkin[2]
[1]Department of Computer Science,
St.Petersburg Forest Technical Academy,
Institutski per. 5, 194021, St.Petersburg, Russia
e-mail: lvu@utkin.usr.etu.spb.ru
[2]Department of Civil Engineering,
Vologda State Technical University,
Mira str., 160000, Vologda, Russia

A new approach to structural reliability analysis under incomplete information about the stress and strength in the form of intervals provided by experts or obtained as a result of imprecise measurements is proposed in the paper. The approach takes into account the possible fact that the number of judgments or observations may be rather small. It is shown that interval-valued information about the stress and strength can be modelled by means of a set of Walley's imprecise Dirichlet models. Very simple expressions for computing bounds for the structural reliability and unreliability are obtained in the paper under conditions of independence and lack of knowledge about independence of the stress and strength. Numerical examples illustrate some advantages of the proposed approach.

Keywords: structural reliability, expert judgments, imprecise probabilities, multinomial model, Dirichlet distribution, random set theory, stress, strength

1. Introduction

A probabilistic model of *structural reliability* and safety has been introduced by Freudenthal [1]. Following his work, a number of studies have been carried out to compute the probability of failure under different assumptions about initial information. Briefly the problem of structural reliability can be stated as follows [2]. Let Y represent a random variable describing the *strength* of a system and let X represent a random variable describing the *stress* or load placed on the system. By assuming that X and Y are defined on $\Omega = [x_{\min}, x_{\max}]$ and $\Theta = [y_{\min}, y_{\max}]$, respectively, system failure occurs when the stress on the system exceeds the strength of the system: $\Phi = \{(x \in \Omega, y \in \Theta) : x \geq y\}$. Here Φ is a region where the combination of system parameters leads to an unacceptable or unsafe system response. Then the *reliability* of the system is determined as $R = \Pr\{X \leq Y\}$, and the *unreliability* is determined as $U = \Pr\{X > Y\} = 1 - R$.

Uncertainty of parameters in engineering design was successfully modelled by means of *interval analysis* [3,4]. Several authors [5–8] used the *fuzzy set* and *possibility* theories [9] to cope with a lack of complete statistical information about the stress and strength. The main idea of their approaches is to consider the stress and strength as fuzzy variables [10] or fuzzy random variables [11]. Authors argued that the assessment of structural parameters is both objective and subjective in character and the best way for describing the subjective component is fuzzy sets. The approach based on using the fuzzy random variables leads to uncertain probability densities and probability distributions, uncertain limit state functions and, as a result of reliability analysis, to fuzzy values for the failure probability and the reliability index. Another approach to structural reliability analysis based on using the *random set* and *evidence* theories [12] has been proposed in [13–15]. Several structural problems solved by

*Corresponding author

means of random set theory have been considered in [16–18]. The random set theory provides us with an appropriate mathematical model of uncertainty when the information about the stress and strength is not complete or when the result of each observation is not point-valued but set-valued, so that it is not possible to assume existence of a unique probability measure.

A more general approach to the structural reliability analysis was proposed in [19–21]. This approach allows us to utilize a wider class of partial information about structural parameters, which includes possible data about probabilities of arbitrary events, expectations of the random stress and strength and their functions. Comparative judgements, for example, the mean value of the stress is less than the mean value of the strength, information about independence or a lack of knowledge about independence of the random stress and strength can be also incorporated in a framework of this approach. At the same time, this approach allows us to avoid additional assumptions about probability distributions of the random parameters because the identification of precise probability distributions requires more information than what experts or deficient statistical data are able to supply. Some extension of the approach was considered in [22]. The main idea proposed in [20,21] is to use *imprecise probability theory* (also called the theory of lower previsions [23], the theory of interval statistical models [24], the theory of interval probabilities [25]), whose general framework is provided by upper and lower previsions (expectations). They can model a very wide variety of kinds of uncertainty, partial information, and ignorance. In other words, the available information about the stress and strength is represented as a set of lower and upper previsions. At the same time, this approach requires to have judgements about probabilistic characteristics of the strength and stress, for example, probabilities, expectations, moments. Unfortunately, we do not have such characteristics in many cases and have only judgments or measurements (observations) of values of the stress and strength themselves. Therefore, the first question is how to utilize the available information and to compute the structural reliability. An answer to this question can be partially found in [16]. The second question is what to do if the number of judgments or measurements is very small.

Therefore, the main aim of the paper is to develop an approach for computing the structural reliability based on expert judgments or interval measurements taking into account the fact that the number of judgments or observations may be rather small. This approach is based on using the *imprecise Dirichlet model* [26]. It is shown in the paper that interval-valued information about the stress and strength can be modelled by means of a set of the imprecise Dirichlet models. This is very interesting and important fact which allows us to use all advantages of the imprecise Dirichlet models in reliability analysis.

The paper is organized as follows. A set of multinomial models produced by interval-valued information about the stress and strength is considered in Section 2. Basic definitions, virtues and shortcomings of Dirichlet distributions and Walley's imprecise model are described in Section 3. Rather simple expressions for lower and upper probabilities of arbitrary events computed with using sets of Walley's imprecise models are obtained in Section 4. In Section 5, results of Section 4 are applied to computing the structural reliability and unreliability under conditions of independence and lack of knowledge about independence. Important special cases such as reliability estimate of a deterministic structure and processing point-valued statistical data are investigated in Section 6. Numerical examples illustrating the different aspects of the proposed approach are given in Section 7.

2. Multinomial models produced by expert judgments and measurements

In this section, we are going to show how a set of expert interval-valued judgments or interval-valued measurements can be represented as a set of multinomial models.

Suppose that there are a_1 intervals A_1, a_2 intervals A_2,..., a_n intervals A_n of the stress X defined on Ω and d_1 intervals D_1, d_2 intervals D_2,..., d_m intervals D_m of the strength Y defined on Θ. All intervals are provided by experts or resulted from measurement. At that, $A_i \subseteq \Omega$ and $D_i \subseteq \Theta$. If all intervals are different, then $a_1 = ... = a_n = 1$ and $d_1 = ... = d_m = 1$. Denote $N = \sum_{i=1}^{n} a_i$ and $M = \sum_{i=1}^{m} d_i$. We assume for simplicity that $\Theta = \Omega$.

Let $\{\mathbf{i}\} = \{(i_1, ..., i_n, i_{n+1})\}$ be a set of all binary vectors consisting of $n+1$ components such that $i_j \in \{0,1\}$. For every vector $\mathbf{i}$, we define the interval B_k ($k = 1, ..., 2^{n-1}$) of the stress as follows:

$$B_k = \left(\bigcap_{j:i_j=1} A_j \right) \bigcap \left(\bigcap_{j:i_j=0} A_j^c \right), \ i_j \in \mathbf{i}.$$

Here $A_{n+1} = \Omega$. As a result, we divide the set Ω into a set of non-intersecting intervals B_k such that $B_1 \cup ... \cup B_L = \Omega$, $L = 2^{n-1}$. Moreover, every interval A_i can be represented as the union of a finite number, l_i, of intervals B_k. The same can be said about intervals of the strength.

For example, suppose that experts supply 1 interval $A_1 = [6, 14]$, 2 intervals $A_2 = [4, 9]$, and 1 interval $A_4 = [2, 11]$ of the stress X defined on $\Omega = [0, 100]$. Here $n = 3$, $a_1 = 1$, $a_2 = 2$, $a_3 = 1$. Intervals B_k are of the form: $B_2 = [2, 4]$, $B_3 = [4, 6]$, $B_4 = [6, 9]$, $B_5 = [9, 11]$, $B_6 = [11, 14]$, and $B_1 \cup B_7 = [0, 2] \cup [14, 100]$.

All points inside the interval B_k are equivalent, that is, we can not assign different probabilities to these points. Therefore, the continues sample space Ω (or Θ) is reduced to a discrete sample space Ω^* (or Θ^*) with L elements.

Associate the interval A_i with an oblong box of size l_i with one open side and the set Ω with L small empty boxes $B_1, ..., B_L$ of size 1. The i-th oblong box contains a_i balls which can move inside the box and we do not know location of balls in the i-th box because its open side is behind. Then we cover small boxes by the i-th oblong box and a_i balls enter in l_i small boxes with numbers from a set J_i. We do not know exact location of balls, but we know that they are in boxes with numbers from J_i. The same procedure is repeated n times. What can we say about possible numbers of balls in the small boxes now? It is obvious that there exist different combinations of numbers of balls except the case when $l_i = 1$ for all $i = 1, ..., n$, i.e., all sets A_i consist of one element of the reduced sample space Ω^*. Suppose that the number of the possible combinations is K. Denote the k-th possible vector of balls by $\mathbf{n}^{(k)} = (n_1^{(k)}, ..., n_L^{(k)})$, $k = 1, ..., K$. If to assume that the sets A_i occurred independently and a ball in the i-th small box has some unknown probability π_i, then every combination of balls in small boxes produces the *standard multinomial model*. K possible combinations of balls produce K equivalent standard multinomial models. The models are equivalent in the sense that we can not choose one of them as a more preferable case.

Thus, the set of intervals $\{A_1, ..., A_n\}$ produces a set of K multinomial models. For every model, the probability of an arbitrary event $A \subseteq \Omega$ depends on $\mathbf{n}^{(k)}$, that is, we can find $P(A|\mathbf{n}^{(k)})$. So far as all the models are equivalent, even by precise probabilities of all categories only lower and upper probabilities of A can be computed as follows:

$$\underline{P}(A) = \min_{k=1,...,K} P(A|\mathbf{n}^{(k)}),$$

$$\overline{P}(A) = \max_{k=1,...,K} P(A|\mathbf{n}^{(k)}).$$

In particular, if all sets A_i consist of single elements, that is, all oblong boxes are of size 1, then $K = 1$ and

$$\underline{P}(A) = P(A|\mathbf{n}^{(1)}), \quad \overline{P}(A) = P(A|\mathbf{n}^{(1)}).$$

The same can be said for the strength. The set of intervals $\{D_1, ..., D_m\}$ produces a set of K^* multinomial models. For every model, the probability of an arbitrary event $D \subseteq \Theta$ depends on vectors of balls $\mathbf{m}^{(k)}$ and lower and upper probabilities of D can be computed as follows:

$$\underline{P}(D) = \min_{k=1,...,K^*} P(D|\mathbf{m}^{(k)}),$$

$$\overline{P}(D) = \max_{k=1,...,K^*} P(D|\mathbf{m}^{(k)}).$$

The following problem is to define $P(A|\mathbf{n}^{(k)})$ and $P(D|\mathbf{m}^{(k)})$. In the case of multinomial samples, the Dirichlet distribution is the traditional choice.

3. Walley's imprecise Dirichlet model

The *Dirichlet* (s,α) *prior distribution* for π, where $\alpha=(\alpha_1,...,\alpha_L)$, has probability density function [27,28]

$$p(\pi)=C(s,\alpha)\cdot\prod_{j=1}^{L}\pi_j^{s\alpha_j-1},$$

where $s>0$, $0<\alpha_j<1$ for $j=1,...,L$, $\alpha\in S(1,L)$, and the proportionality constant C is determined by the fact that the integral of $p(\pi)$ over the simplex of possible values of π is 1 and

$$C(s,\alpha)=\Gamma(s)\left(\prod_{j=1}^{L}\Gamma(s\alpha_j)\right)^{-1}.$$

Here α_i is the mean of π_i under the Dirichlet prior and s determines the influence of the prior distribution on posterior probabilities. $\Gamma(\cdot)$ is the Gamma-function which satisfies $\Gamma(x+1)=x\Gamma(x)$ and $\Gamma(1)=1$. $S(1,L)$ denotes the interior of the unit simplex.

Walley [26] pointed out several reasons for using a set of Dirichlet distributions to model prior ignorance about probabilities π:

1. Dirichlet prior distributions are mathematically tractable because they generate Dirichlet posterior distributions;

2. sets of Dirichlet distributions are very rich, because they produce the same inferences as their convex hulls and any prior distribution can be approximated by a finite mixture of Dirichlet distributions;

3. the most common Bayesian models for prior ignorance about probabilities π are Dirichlet distributions.

   The *imprecise Dirichlet model* is defined by Walley [26] as the set of all Dirichlet (s,α) distributions such that $\alpha\in S(1,L)$.

For the imprecise Dirichlet model, the *hyperparameter* s determines how quickly upper and lower probabilities of events converge as statistical data accumulate. Walley [26] defined s as a number of observations needed to reduce the imprecision (difference between upper and lower probabilities) to half its initial value. Smaller values of s produce faster convergence and stronger conclusions, whereas large values of s produce more cautious inferences. At the same time, the value of s must not depend on L or a number of observations. The detailed discussion concerning the parameter s and the imprecise Dirichlet model can be found in [29–32,26]. The application of the model in reliability was studied in [31]. This model was also applied to the game theory for choosing a strategy in a two-player game by Quaeghebeur and de Cooman [33]. An approach to dealing with incomplete sets of multivariate categorical data by exploiting Walley's imprecise Dirichlet model was studied by Zaffalon [34].

Let A be any non-trivial subset of a sample space $\{\omega_1,...,\omega_L\}$, and let $n(A)$ denote the observed number of occurrences of A in the N trials, $n(A)=\sum_{\omega_j\in A}n_j$. Then, according to [26], the predictive probability $P(A,s)$ under the Dirichlet posterior distribution is

$$P(A,s)=\frac{n(A)+s\alpha(A)}{N+s},$$

where $\alpha(A)=\sum_{\omega_j\in A}\alpha_j$.

By maximizing and minimizing α_j under restriction $\alpha\in S(1,L)$, we obtain the posterior upper and lower probabilities of A:

$$\underline{P}(A,s)=\frac{n(A)}{N+s},\ \overline{P}(A,s)=\frac{n(A)+s}{N+s}.$$

By returning to the multinomial models considered in the example with boxes and balls and assuming that probabilities of balls are governed by the Dirichlet distribution, we can write the lower $\underline{P}(A,s)$ and upper $\overline{P}(A,s)$ probabilities of an event A covering small boxes (B_i) with numbers from the set J as follows:

$$\underline{P}(A,s)=\min_{k=1,...,K}\inf_{\alpha\in S(1,L)}\frac{n^{(k)}(A)+s\alpha(A)}{N+s},$$

$$\overline{P}(A,s)=\max_{k=1,...,K}\sup_{\alpha\in S(1,L)}\frac{n^{(k)}(A)+s\alpha(A)}{N+s},$$

where

$$\alpha(A) = \sum_{j\in J} \alpha_j, \quad n^{(k)}(A) = \sum_{j\in J} n_j^{(k)}, \ N = \sum_{i=1}^{n} a_i.$$

4. Computing lower and upper probabilities of events

The general expressions for lower and upper probabilities of an event in the form of optimization problems have been obtained in the previous section. Now our aim is to solve these problems in order to get computationally simple expressions.

The lower and upper probabilities $\underline{P}(A,s)$ and $\overline{P}(A,s)$ can be rewritten as

$$\underline{P}(A,s) = \frac{\min\limits_{k=1,\dots,K} n^{(k)}(A) + s \cdot \inf\limits_{\alpha\in S(1,L)} \alpha(A)}{N+s},$$

$$\overline{P}(A,s) = \frac{\max\limits_{k=1,\dots,K} n^{(k)}(A) + s \cdot \sup\limits_{\alpha\in S(1,L)} \alpha(A)}{N+s}.$$

Note that $\inf_{\alpha\in S(1,L)} \alpha(A)$ is achieved at $\alpha(A) = 0$ and $\sup_{\alpha\in S(1,L)} \alpha(A)$ is achieved at $\alpha(A) = 1$ except a case when $A = \Omega$. If $A = \Omega$, then $\alpha(A) = 1$ for the minimum and maximum.

In order to find the minimum and maximum of $n^{(k)}(A)$ we consider three sets A_1, A_2, A_3 such that $A_1 \subseteq A$, $A_2 \cap A = \varnothing$, $A_3 \cap A \neq \varnothing$ and $A_3 \subsetneq A$. Numbers of their occurrences are a_1, a_2, a_3, respectively. It is obvious that all balls (a_1) corresponding to the set A_1 belong to the set A and $n^{(k)}(A)$ can not be less than a_1. On the other hand, all balls (a_2) corresponding to the set A_2 do not belong to A. This implies that $n^{(k)}(A)$ can not be greater than $N - a_2$. A part of balls corresponding to A_3 may belong to A, but it is not necessary. Therefore, $\min n^{(k)}(A) = a_1$ and $\max n^{(k)}(A) = N - a_2$. By extending this reasoning on an arbitrary set of A_i, we get the minimal, $L_1(A)$, and maximal, $L_2(A)$, values of $n^{(k)}(A)$:

$$L_1(A) = \min_{k=1,\dots,K} n^{(k)}(A) = \sum_{i:A_i\subseteq A} a_i,$$

$$\begin{aligned} L_2(A) &= \max_{k=1,\dots,K} n^{(k)}(A) \\ &= N - \sum_{i:A_i\cap A=\varnothing} a_i = \sum_{i:A_i\cap A\neq\varnothing} a_i. \end{aligned}$$

Hence

$$\underline{P}(A,s) = \frac{L_1(A)}{N+s},$$

$$\overline{P}(A,s) = \frac{L_2(A)+s}{N+s}.$$

We have obtain very simple expressions for lower and upper probabilities for any interval A. Similar results can be produced for probabilities of an interval of the strength.

5. Structural reliability and unreliability

Since we know how to find probabilities of any intervals, then the probability distribution functions of the stress $F(x,s) = \Pr(X \leq x, s)$ and the strength $Q(y,s) = \Pr(Y \leq y, s)$ can be computed as probabilities of intervals $A = [x_{\min}, x]$ and $D = [y_{\min}, y]$, respectively. It is obvious that interval initial data produce lower and upper probability distribution functions of the stress $\underline{F}$, $\overline{F}$ and the strength $\underline{Q}$, $\overline{Q}$. By using the results of the previous section and taking into account that $A_i \subseteq [x_{\min}, x]$ if $\sup A_i \leq x$, and $A_i \cap [x_{\min}, x] \neq \varnothing$ if $\inf A_i \leq x$, $\alpha(\Omega) = 1$ and $\alpha(\Theta) = 1$, we get

$$\underline{F}(x,s) = \begin{cases} (N+s)^{-1} \sum\limits_{i:\sup A_i\leq x} a_i, & x < x_{\max} \\ 1, & x = x_{\max} \end{cases},$$

$$\overline{F}(x,s) = \begin{cases} (N+s)^{-1} \left(s + \sum\limits_{i:\inf A_i\leq x} a_i \right), & x > x_{\min} \\ 0, & x = x_{\min} \end{cases}.$$

Similarly, we obtain the lower and upper probability distribution functions of the strength

$$\underline{Q}(y,s) = \begin{cases} (M+s)^{-1} \sum\limits_{i:\sup D_i\leq y} d_i, & y < y_{\max} \\ 1, & y = y_{\max} \end{cases},$$

$$\overline{Q}(y,s) = \begin{cases} (M+s)^{-1}\left(s+\sum_{i:\inf D_i\le y} d_i\right), & y>y_{\min} \\ 0, & y=y_{\min} \end{cases}.$$

5.1. Independent stress and strength

Let us consider a case when random variables corresponding to the stress and the strength are statistically independent. According to [20,21], the lower $\underline{U}(s)$ and upper $\overline{U}(s)$ structural unreliabilities by known lower $\underline{F}$, $\underline{Q}$ and upper $\overline{F}$, $\overline{Q}$ probability functions of the stress and strength as functions of the hyperparameter s can be found as

$$\underline{U}(s) = \int_\Omega \underline{q}(y,s)(1-\overline{F}(y,s))\mathrm{d}y,$$

$$\overline{U}(s) = 1-\int_\Omega \underline{f}(x,s)(1-\overline{Q}(x,s))\mathrm{d}x,$$

where $\underline{q}$ and $\underline{f}$ are lower probability density functions of the strength and the stress, respectively, defined as

$$\underline{q}(x,s) = \lim_{\Delta x\to 0}\frac{\underline{Q}(x+\Delta x,s)-\underline{Q}(x,s)}{\Delta x},$$

$$\underline{f}(x,s) = \lim_{\Delta x\to 0}\frac{\underline{F}(x+\Delta x,s)-\underline{F}(x,s)}{\Delta x}.$$

Note that the distribution functions $\underline{F}$ and $\underline{Q}$ are step functions. This implies that the considered density functions are of the form:

$$\underline{q}(x,s) = (M+s)^{-1}\sum_{i:\sup D_i\le x} d_i\delta(x-\sup D_i),$$

$$\underline{f}(x,s) = (N+s)^{-1}\sum_{i:\sup A_i\le x} a_i\delta(x-\sup A_i),$$

where $\delta(x-a)$ is Dirac function which has unit area concentrated in the immediate vicinity of some point a.

By substituting the obtained densities into expressions for $\underline{U}(s)$ and $\overline{U}(s)$, we obtain

$$\underline{U}(s) = (M+s)^{-1} \times \int_\Omega\left(\sum_{i:\sup D_i\le x} d_i\delta(x-\sup D_i)\right) \times\left(1-(N+s)^{-1}\left(s-\sum_{i:\inf A_i\le x} a_i\right)\right)\mathrm{d}x,$$

$$\overline{U}(s) = 1-(N+s)^{-1} \times \int_\Omega\left(\sum_{i:\sup A_i\le x} a_i\delta(x-\sup A_i)\right) \times\left(1-(M+s)^{-1}\left(s-\sum_{i:\inf D_i\le x} d_i\right)\right)\mathrm{d}x.$$

Hence, there hold

$$\underline{U}(s) = (M+s)^{-1}(N+s)^{-1} \times\sum_{i=1}^{m} d_i\left(N-\sum_{j:\inf A_j\le\sup D_i} a_j\right), \quad (1)$$

$$\overline{U}(s) = 1-(M+s)^{-1}(N+s)^{-1} \times\sum_{i=1}^{n} a_i\left(M-\sum_{j:\inf D_j\le\sup A_i} d_j\right). \quad (2)$$

Thus, we have obtained simple expressions for computing the lower and upper structural unreliabilities. Structural reliabilities can be found from the above bounds as follows: $\underline{R}(s) = 1-\overline{U}(s)$, $\overline{R}(s) = 1-\underline{U}(s)$.

Let us consider a special case $s=0$. If we denote $m_i^{(X)} = a_i/N$ and $m_i^{(Y)} = d_i/M$, then the lower and upper unreliabilities can be written as follows:

$$\underline{U}(0) = \sum_{i=1}^{m} m_i^{(Y)} \left(1 - \sum_{j:\inf A_j \leq \sup D_i} m_j^{(X)}\right),$$

$$\overline{U}(0) = 1 - \sum_{i=1}^{n} m_i^{(X)} \left(1 - \sum_{j:\inf D_j \leq \sup A_i} m_j^{(Y)}\right).$$

Values $m_i^{(X)}$ and $m_i^{(Y)}$ are none other than the *basic probability assignments* defined in *random set theory* [35–37]. This implies that $\underline{U}(0)$ and $\overline{U}(0)$ in the special case $s = 0$ are the *belief* Bel and *plausibility* Pl measures of an event $X > Y$. Thus, the proposed model can be regarded as the extension of models [16] based on random set theory, which allows us to take into account the lack of sufficient statistical data and cases where probability functions that govern the random processes are not exactly known.

5.2. Lack of knowledge about independence

It is also worth noticing that the problem of computing the structural reliability can be solved without having to introduce an assumption about the independence of variables X and Y. This does not mean that the stress and strength are dependent. Just the analyst may be ignorant of whether the variables are dependent or not. The asterisk notation in $\underline{U}^*(s)$ ($\underline{R}^*(s)$) and $\overline{U}^*(s)$ ($\overline{R}^*(s)$) will mean that the bounds for the structural unreliability (reliability) are obtained based on ignorance about the dependency of X and Y. According to [20,21], there hold

$$\underline{U}^*(s) = \max_{z\geq 0} \max\left\{0, \underline{Q}(z,s) - \overline{F}(z,s)\right\},$$

$$\overline{U}^*(s) = 1 - \max_{z\geq 0} \max\left\{0, \underline{F}(z,s) - \overline{Q}(z,s)\right\}.$$

By substituting the expressions for the probability distribution functions into the above formulae, we get

$$\underline{U}^*(s) = \max_{z\geq 0} \max\left\{0, \frac{1}{M+s} \sum_{i:\sup D_i \leq z} d_i - \frac{s}{N+s} - \frac{1}{N+s} \sum_{i:\inf A_i \leq z} a_i \right\},$$

$$\overline{U}^*(s) = 1 - \max_{z\geq 0} \max\left\{0, \frac{1}{N+s} \sum_{i:\sup A_i \leq z} a_i - \frac{s}{M+s} - \frac{1}{M+s} \sum_{i:\inf D_i \leq z} d_i \right\}.$$

Since the distribution functions are step functions, then the last expressions can be simplified

$$\underline{U}^*(s) = \max_{i=1,\dots,m} \max\left\{0, \frac{1}{M+s} \sum_{j=1}^{i} d_j - \frac{s}{N+s} - \frac{1}{N+s} \sum_{j:\inf A_j \leq \sup D_i} a_j \right\}, \quad (3)$$

$$\overline{U}^*(s) = 1 - \max_{i=1,\dots,n} \max\left\{0, \frac{1}{N+s} \sum_{j=1}^{i} a_j - \frac{s}{M+s} - \frac{1}{M+s} \sum_{j:\inf D_j \leq \sup A_i} d_j \right\}. \quad (4)$$

If $s = 0$ and $m_i^{(X)} = a_i/N$, $m_i^{(Y)} = d_i/M$, then there hold

$$\underline{U}^*(0) = \max_{i=1,\dots,m} \max \times \left\{0, \sum_{j=1}^{i} m_j^{(Y)} - \sum_{j:\inf A_j \leq \sup D_i} m_j^{(X)} \right\},$$

$$\overline{U}^*(0) = 1 - \max_{i=1,\dots,n} \max \times \left\{0, \sum_{j=1}^{i} m_j^{(X)} - \sum_{j:\inf D_j \leq \sup A_i} m_j^{(Y)} \right\}.$$

It can simply be proved that $[\underline{U}(s), \overline{U}(s)] \subseteq [\underline{U}^*(s), \overline{U}^*(s)]$ for all $s \geq 0$. This is obvious because the independence condition is some additional information reducing the imprecision of resulting measures.

6. Important special cases

6.1. Reliability estimate of a deterministic structure

Very often the problem of structural reliability analysis consists of checking the requirement that the stress does not exceed (or exceeds) the specified limit value y_0 with some probability. For example, a bar of uniform cross-sectional area, g, is subjected to a load, X (see Tonon at al. [16]). We are interested in checking if the axial stress, X/g, does not exceed the specified stress σ with some probability. Then the reliability is determined as follows: $R = \Pr\{X \leq g \cdot \sigma\}$.

Formally, in this special case the variable X takes a predefined value y_0 ($g \cdot \sigma$ in the example) with probability 1 and the reliability (unreliability) is defined as $R = \Pr\{X \leq y_0\}$ ($U = \Pr\{X \geq y_0\}$). Let us show that the obtained expressions can be simply modified in order to solve this problem. Indeed, there holds $R(s) = \Pr\{X \leq y_0\} = F(y_0, s)$, that is, the reliability is none other than the value of the probability distribution function of the stress at the point y_0. Consequently, if $x_{\min} < y_0 < x_{\max}$, then there hold

$$\underline{R}(s) = \frac{1}{N+s} \sum_{i:\sup A_i \leq x} a_i,$$

$$\overline{R}(s) = \frac{s}{N+s} + \frac{1}{N+s} \sum_{i:\inf A_i \leq x} a_i,$$

$$\underline{U}(s) = 1 - \frac{s}{N+s} - \frac{1}{N+s} \sum_{i:\inf A_i \leq x} a_i,$$

$$\overline{U}(s) = 1 - \frac{1}{N+s} \sum_{i:\sup A_i \leq x} a_i.$$

In particular, if $s = 0$, then we have

$$\underline{U}(0) = 1 - \sum_{i:\inf A_i \leq x} m_i^{(X)},$$

$$\overline{U}(0) = 1 - \sum_{i:\sup A_i \leq x} m_i^{(X)}.$$

The bounds for this special case ($s = 0$) have been presented by Tonon at al. [16].

6.2. Point-valued statistical data

Suppose that there are available sets of point-valued statistical data related to the stress and strength. Let $x_1, ..., x_N$ and $y_1, ..., y_M$ be realizations of random variables X and Y, respectively. For simplicity, we assume that $x_1 < ... < x_N$ and $y_1 < ... < y_M$. Introduce the following intervals: $A_i = [x_i - \varepsilon, x_i + \varepsilon]$, $i = 1, ..., N$, and $D_i = [y_i - \varepsilon, y_i + \varepsilon]$, $i = 1, ..., M$. Here ε is a real number which, for instance, can be regarded as the measurement error. Thus, we have sets of intervals of the stress and strength. Therefore, the obtained results can be applied. Note that $a_i = 1$ and $d_j = 1$ for all $i = 1, ..., N$ and $j = 1, ..., M$ in this case. If the stress and strength are statistically independent and $\inf A_i = x_i - \varepsilon$, $\sup A_i = x_i + \varepsilon$, $\inf D_i = y_i - \varepsilon$, $\sup D_i = y_i + \varepsilon$, then Eqs. (1) and (2) can be rewritten as follows:

$$\underline{U}(s) = (M+s)^{-1}(N+s)^{-1} \times \left(MN - \sum_{i=1}^{M} \left(\sum_{j:x_j - \varepsilon \leq y_i + \varepsilon} 1 \right) \right),$$

$$\overline{U}(s) = 1 - (M+s)^{-1}(N+s)^{-1} \times \left(MN - \sum_{i=1}^{N} \left(\sum_{j:y_j - \varepsilon \leq x_i + \varepsilon} 1 \right) \right).$$

Denote the number of points x_j such that $x_j \leq y_i$ by $n(i)$, and the number of points y_j such that $y_j \leq x_i$ by $m(i)$. If $\varepsilon \to 0$, then

$$\underline{U}(s) = \frac{1}{(M+s)(N+s)} \times \left(MN - \sum_{i=1}^{M} n(i) \right), \quad (5)$$

$$\overline{U}(s) = 1 - \frac{1}{(M+s)(N+s)} \times \left(MN - \sum_{i=1}^{N} m(i) \right). \quad (6)$$

If there is no information about independence of the stress and strength, then we can similarly write

$$\underline{U}^*(s) = \max_{i=1,\dots,m} \max \times \left\{0, \frac{i}{M+s} - \frac{s+n(i)}{N+s}\right\}, \quad (7)$$

$$\overline{U}^*(s) = 1 - \max_{i=1,\dots,n} \max \times \left\{0, \frac{i}{N+s} - \frac{s+m(i)}{M+s}\right\}. \quad (8)$$

7. Numerical examples

7.1. Example 1

Suppose that an expert supplies one interval-valued judgment $A_1 = [1,5]$ ($N = 1$) about the stress and one interval-valued judgment $D_1 = [6,10]$ ($M = 1$), about the strength. According to Eqs. (1) and (2), there hold

$$\underline{U}(s) = (1+s)^{-1}(1+s)^{-1}(1-1) = 0,$$

$$\overline{U}(s) = 1 - (1+s)^{-1}(1+s)^{-1}(1-0).$$

If to assume that $s = 0$, then we obtain $\underline{U}(0) = 0$ and $\overline{U}(0) = 0$. It is obvious that the conclusion based on single judgments is mistaken. However, if we take, for instance, $s = 1$, then $\underline{U}(1) = 0$ and $\overline{U}(1) = 0.75$. This is a more believable result, and its imprecision reflects the lack of sufficient statistical data.

7.2. Example 2

Suppose that $N = 100$ intervals of the stress and $M = 100$ intervals of the strength were supplied by experts. At that, it is assumed that the stress and strength are independent. These intervals are given in Table 1. If we take $s = 0$, then, by using Eqs. (1) and (2), we get $\underline{U}(0) = 0$, $\overline{U}(0) = 0.17$. If $s = 1$, then $\underline{U}(1) = 0$, $\overline{U}(1) = 0.19$. It can be seen from these results that the bounds slightly differ because the number of judgments is rather large and the hyperparameter s does not impact on the bounds significantly.

Let us take only $N = 6$ the same intervals of the stress and $M = 5$ the same intervals of the strength approximately in the same proportion.

Table 1
100 intervals of the stress and 100 intervals of the strength supplied by experts

Stress			Strength		
i	a_i	A_i	i	d_i	D_i
1	17	$[10,15]$	1	25	$[16,22]$
2	59	$[12,13]$	2	13	$[18,20]$
3	24	$[12,17]$	3	27	$[14,18]$
			4	35	$[20,24]$

These intervals are given in Table 2. It is assumed that they are a part of 200 intervals supplied by the experts in the previous case. If $s = 0$, then $\underline{U}(0) = 0$, $\overline{U}(0) = 0.1$. This result shows that it is very incautious because $\overline{U}(0) = 0.1 < 0.17$. How to get more cautious bounds for the stress-strength unreliability? It is necessary to accept $s = 1$. Then there hold $\underline{U}(1) = 0$, $\overline{U}(1) = 0.36$. It is worth noticing that $\overline{U}(1) = 0.36 > 0.17$.

Now suppose that there is no information about independence of the stress and strength. Then, by using Eqs. (3) and (4), we get for the case $N = 100$, $M = 100$, $s = 1$ the following: $\underline{U}^*(1) = 0$, $\overline{U}^*(1) = 0.35$. If $N = 6$, $M = 5$, $s = 1$, then $\underline{U}^*(1) = 0$, $\overline{U}^*(1) = 0.45$. It is obvious that $[\underline{U}(s), \overline{U}(s)] \subseteq [\underline{U}^*(s), \overline{U}^*(s)]$ because the independence can be regarded as some additional information.

Suppose that 2 intervals $[12,13]$ of the stress and 1 interval $[20,24]$ of the strength have been obtained after additional elicitation. Then $N = 8$, $a_2 = 6$, $M = 6$, $d_4 = 3$. Hence, we update the unreliabilities $\underline{U}(1) = 0$, $\overline{U}(1) = 0.28$. One can see that more precise results have been obtained after additional judgments.

7.3. Example 3

Let us consider point-valued statistical data. Suppose that the stress and strength are governed by the exponential distribution. At that the mean value of the stress is $T_{stress} = 2$ and the mean value of the strength is $T_{strength} = 9$. Then by using the well-known expression for the unreliability in this case, we compute the unreliability

$$U = \frac{T_{stress}}{T_{stress} + T_{strength}} = \frac{2}{2+9} = 0.182.$$

Table 2
6 intervals of the stress and 5 intervals of the strength supplied by experts

Stress			Strength		
i	a_i	A_i	i	d_i	D_i
1	1	$[10, 15]$	1	1	$[16, 22]$
2	4	$[12, 13]$	2	1	$[18, 20]$
3	1	$[12, 17]$	3	1	$[14, 18]$
			4	2	$[20, 24]$

In order to illustrate the proposed approach, we generate $N = 6$ random numbers corresponding to the stress and $M = 5$ random numbers corresponding to the strength. The results of generation are

0.42, 1.32, 6. 33, 3.398, 2.1, 2.3

for the stress and

2.57, 7.04, 11.4, 17.7, 3.97

for the strength. Now we assume that the probability distributions of the stress and strength are unknown, and there are available only the above sample values. By using Eqs. (5) and (6), we get

$$\begin{aligned}\underline{U}(s) &= \frac{1}{(5+s)(6+s)} \\ &\quad \times (5 \cdot 6 - 4 - 6 - 6 - 6 - 5) \\ &= \frac{3}{(5+s)(6+s)},\end{aligned}$$

$$\begin{aligned}\overline{U}(s) &= 1 - \frac{1}{(5+s)(6+s)} \\ &\quad \times (5 \cdot 6 - 0 - 0 - 2 - 1 - 0 - 0) \\ &= 1 - \frac{27}{(5+s)(6+s)}.\end{aligned}$$

If $s = 0$, then $\underline{U}(0) = \overline{U}(0) = 0.1$. One can see that this value is substantially smaller than $U = 0.182$. However, if we take $s = 1$, then $\underline{U}(1) = 0.071$ and $\overline{U}(1) = 0.357$. Hence, there holds $\underline{U}(1) \leq U \leq \overline{U}(1)$. Despite the imprecision of the obtained unreliability by $s = 1$, we have cautious bounds for the "true" unreliability 0.182.

It is worth noticing that cautious bounds in this example take place if $s = 0.3$. In this case, $\underline{U}(0.3) = 0.09$ and $\overline{U}(0.3) = 0.191$. Unfortunately, we can find "minimal value" of the hyperparameter s providing the minimal cautious bounds for the unreliability only by having the "true" unreliability.

8. Conclusion

The very simple expressions for computing bounds for the structural reliability and unreliability have been obtained in the paper. The following advantages of the proposed approach can be noted:

1. The computation of the structural reliability is independent of the definition of the sample space of the stress and strength. This is due to the special forming of the set of elementary intervals $B_1, ..., B_L$ (see Section 2) and due to the representation invariance principle considered in detail by Walley [26].

2. The proposed approach can be regarded as the extension of methods based on random set theory. This extension may allow us to avoid some unlikely results (see Subsection 7.1) and to get more cautious belief and plausibility measures of structural reliability.

3. The approach allows us to obtain more realistic structural reliability measures for incomplete and imprecise data.

4. The approach can take into account the imprecision due to the missing data. In this case, the hyperparameter s is interpreted as the number of hidden or missing observations.

5. The structural reliability measures can simply be updated after observing new events (observations). This is due to the property of Dirichlet prior distributions to generate Dirichlet posterior distributions.

6. Point-valued statistical data also can be incorporated into the considered framework (see Subsection 6.2).

At the same time, we have to point out that the proper choice of the hyperparameter s is an open question and further work is needed to find some strong rules for determining s.

REFERENCES

1. A.N. Freudenthal. Safety and the probability of structural failure. *Transactions ASCE*, 121:1337–1397, 1956.
2. V.V. Bolotin. *Statistical Method in Structural Mechanics*. Holden Press, San Francisco, 1969.
3. R. Penmetsa and R.V. Grandhi. Efficient estimation of structural reliability for problems with uncertain intervals. *International Journal of Computers and Structures*, 80:1103–1112, March 2002.
4. L. Wang R.V. Grandhi. Reliability-based structural optimization using improved two-point adaptive non-linear approximations. *Finite Elements in Analysis and Design*, 29:35–48, 1998.
5. A. Bardossy and I. Bogardi. Fuzzy fatigue life prediction. *Structural Safety*, 6:25–38, 1989.
6. B. Möller, M. Beer, W. Graf, and A. Hoffmann. Possibility theory based safety assessment. *Comp.-Aided Civil and Infrastruct. Eng.*, 14:81–91, 1999.
7. R. Penmetsa and R.V. Grandhi. Uncertainty propagation using possibility theory and function approximation. *International Journal of Mechanics of Structures and Machines*, 2001. Submitted for publication.
8. L. Yubin, Q. Zhong, and W. Guangyuan. Fuzzy random reliability of structures based on fuzzy random variables. *Fuzzy Sets and Systems*, 86:345–355, 1997.
9. D. Dubois and H. Prade. *Possibility Theory: An Approach to Computerized Processing of Uncertainty*. Plenum Press, New York, 1988.
10. S. Nahmias. Fuzzy variable. *Fuzzy Sets and Systems*, 1:97–110, 1978.
11. H. Kwakernaak. Fuzzy random variables: definitions and theorems. *Information Sciences*, 15:1–29, 1978.
12. J. Goutsias, R.P.S. Mahler, and H.T. Nguyen. *Random Sets - Theory and Applications*. Springer, New York, 1997.
13. H.-R. Bae, R.V. Grandhi, and R.A. Canfield. Sensitivity analysis of structural response uncertainty propagation using evidence theory. In *Proc. of 9-th AIAA/ISSMO Symposium on Multidisciplinary Analysis and Optimization*, pages 1–11, Atlanta, Georgia, USA, September 2002. AIAA.
14. J. Hall and J. Lawry. Imprecise probabilities of engineering system failure from random and fuzzy set reliability analysis. In G. de Cooman, T.L. Fine, and T. Seidenfeld, editors, *Imprecise Probabilities and Their Applications. Proc. of the 1st Int. Symposium ISIPTA'01*, pages 195–204, Ithaca, USA, June 2001. Shaker Publishing.
15. F. Tonon and A. Bernardini. A random set approach to optimisation of uncertain structures. *Computers and Structures*, 68:583–600, 1998.
16. F. Tonon, A. Bernardini, and I. Elishakoff. Concept of random sets as applied to the design of structures and analysis of expert opinions for aircraft crash. *Chaos, Solutions and Fractals*, 10(11):1855–1868, 1999.
17. F. Tonon, A. Bernardini, and A. Mammino. Determination of parameters range in rock engineering by means of random set theory. *Reliability Engineering and System Safety*, 70:241–261, 3 2000.
18. F. Tonon, A. Bernardini, and A. Mammino. Reliability analysis of rock mass response by means of random set theory. *Reliability Engineering and System Safety*, 70:263–282, 3 2000.
19. L.V. Utkin. A second-order uncertainty model of independent random variables: An example of the stress-strength reliability. In J.-M. Bernard, T. Seidenfeld, and M. Zaffalon, editors, *Proc. of the 3rd Int. Symposium on Imprecise Probabilities and Their Applications, ISIPTA'03*, pages 530–544, Lugano, Switzerland, July 2003. Carleton Scientific.
20. L.V. Utkin and I.O. Kozine. Stress-strength

reliability models under incomplete information. *International Journal of General Systems*, 31(6):549 – 568, 2002.
21. L.V. Utkin and I.O. Kozine. Structural reliability modelling under partial source information. In H. Langseth and B. Lindqvist, editors, *Proceedings of the Third Int. Conf. on Mathematical Methods in Reliability (Methodology and Practice)*, pages 647–650, Trondheim, Norway, June 2002. NTNU.
22. L.V. Utkin. An uncertainty model of the stress-strength reliability with imprecise parameters of probability distributions. *Zeitschrift für Angewandte Mathematik und Mechanik (Applied Mathematics and Mechanics)*, 2004. To Appear.
23. P. Walley. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London, 1991.
24. V. P. Kuznetsov. *Interval Statistical Models*. Radio and Communication, Moscow, 1991. in Russian.
25. K. Weichselberger. *Elementare Grundbegriffe einer allgemeineren Wahrscheinlichkeitsrechnung*, volume I Intervallwahrscheinlichkeit als umfassendes Konzept. Physika, Heidelberg, 2001.
26. P. Walley. Inferences from multinomial data: Learning about a bag of marbles. *Journal of the Royal Statistical Society, Series B*, 58:3Ű57, 1996. with discussion.
27. M. DeGroot. *Optimal Statistical Decisions*. McGraw-Hill, New York, 1970.
28. S. Wilks. *Mathematical Statistics*. Wiley and Sons, New York, 1962.
29. J.-M. Bernard. Non-parametric inference about an unknown mean using the imprecise Dirichlet model. In G. de Cooman, T.L. Fine, and T. Seidenfeld, editors, *Imprecise Probabilities and Their Applications. Proc. of the 2nd Int. Symposium ISIPTA'01*, pages 40–50, Ithaca, USA, June 2001. Shaker Publishing.
30. J.-M. Bernard. Implicative analysis for multivariate binary data using an imprecise Dirichlet model. *Journal of Statistical Planning and Inference*, 105(1):83–104, 2002.
31. F.P.A. Coolen. An imprecise Dirichlet model for Bayesian analysis of failure data including right-censored observations. *Reliability Engineering and System Safety*, 56:61–68, 1997.
32. M. Smithson, T. Bartos, and K. Takemura. Human judgment under sample space ignorance. In G. de Cooman, F.G. Cozman, S. Moral, and P. Walley, editors, *ISIPTA '99 - Proceedings of the First International Symposium on Imprecise Probabilities and Their Applications*, pages 324–332, Zwijnaarde, Belgium, June 1999.
33. E. Quaeghebeur and G. de Cooman. Game-theoretic learning using the imprecise Dirichlet model. In J.-M. Bernard, T. Seidenfeld, and M. Zaffalon, editors, *Proc. of the 3rd Int. Symposium on Imprecise Probabilities and Their Applications, ISIPTA'03*, pages 450–464, Lugano, Switzerland, July 2003. Carleton Scientific.
34. M. Zaffalon. Exact credal treatment of missing data. *Journal of Statistical Planning and Inference*, 105(1):105–122, 2002.
35. A.P. Dempster. Upper and lower probabilities induced by a multi-valued mapping. *Annales of Mathematical Statistics*, 38:325–339, 1967.
36. G.J. Klir and M.J. Wierman. *Uncertainty-Based Information: Elements of Generalized Information Theory*. Physica-Verlag, Heidelberg, 1998.
37. G. Shafer. *A Mathematical Theory of Evidence*. Princeton University Press, 1976.