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An uncertainty model of structural reliability with imprecise parameters of probability distributions

An approach to compute bounds for the structural reliability by imprecise parameters of the stress and strength probability distributions is proposed. The approach is based on using imprecise probability theory and takes into account different types of independence of the stress, strength and their parameters. It is shown that computation of the imprecise stress-strength model can be reduced to solution of a number of linear programming problems. Special cases of the exponentially distributed stress and strength are considered. Various numerical examples illustrate the approach and show the impact of the independence conditions on imprecision of results.

Keywords: structural reliability, expert judgements, imprecise probabilities, natural extension, previsions, linear programming

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1 Introduction

A probabilistic model of structural reliability and safety has been introduced by Freudenthal [12]. Following his work, a number of studies have been carried out to compute the probability of failure under different assumptions about initial information. Briefly the problem of structural reliability can be stated as follows [6]. Let Y represent a random variable describing the strength of a system and let X represent a random variable describing the stress or load placed on the system. By assuming that X and Y are defined on $\mathcal{X}$ and $\mathcal{Y}$, respectively, system failure occurs when the stress on the system exceeds the strength of the system: $\Phi = \{(x \in \mathcal{X}, y \in \mathcal{Y}) : x \geq y\}$. Here Φ is a region where the combination of system parameters leads to an unacceptable or unsafe system response. Then the reliability of the system is determined as $R = \Pr\{X \leq Y\}$, and the unreliability is determined as $Q = \Pr\{X > Y\} = 1 - R$.

Uncertainty of parameters in engineering design was successfully modelled by means of interval analysis [25, 24]. Several authors [4, 21, 23, 35] used the fuzzy set and possibility theories [10] to cope with a lack of complete statistical information about the stress and strength. The main idea of their approaches is to consider the stress and strength as fuzzy variables [22] or fuzzy random variables [20]. Authors argued that the assessment of structural parameters is both objective and subjective in character and the best way for describing the subjective component is fuzzy sets. The approach based on using the fuzzy random variables leads to uncertain probability densities and probability distributions, uncertain limit state functions and, as a result of reliability analysis, to fuzzy values for the failure probability and the reliability index. Another approach to structural reliability analysis based on using the random set and evidence theories [16] has been proposed in [3, 18, 26]. Several structural problems solved by means of random set theory have been considered in [27, 28, 29]. The random set theory provides us with an appropriate mathematical model of uncertainty when the information about the stress and strength is not complete or when the result of each observation is not point-valued but set-valued, so that it is not possible to assume existence of a unique probability measure.

A more general approach to the structural reliability analysis was proposed in [31, 32]. This approach allows us to utilize a wider class of partial information about structural parameters, which includes possible data about probabilities of arbitrary events, expectations of the random stress and strength and their functions. Comparative judgements, for example, the mean value of the stress is less than the mean value of the strength, information about independence or a lack of knowledge about independence of the random stress and strength can be also incorporated in a framework of this approach. At the same time, this approach allows us to avoid additional assumptions about probability distributions of the random parameters because the identification of precise probability distributions requires more information than what experts or deficient statistical data are able to supply. The main idea proposed in [31, 32] is to use imprecise probability theory (also called the theory of lower previsions [33], the theory of interval statistical models [19], the theory of interval probabilities [34]), whose general framework is provided by upper and lower previsions. They can model a very wide variety of kinds of uncertainty, partial information, and ignorance. In other words, the available information about the stress and strength is represented as a set of lower and upper previsions. In order to compute the structural reliability by taking into account the available information, a general procedure called natural extension is used. It produces a coherent overall model [33] from a certain collection of imprecise probability judgements and may be seen as a basic constructive step in imprecise statistical reasoning.

However, there are cases when types of probability distributions of the stress and strength are known, for example, from their physical nature, but parameters or a part of parameters of distributions are defined by experts. If to assume that experts provide possible intervals of parameters and these experts are absolutely reliable, i.e., they provide always true assessments, then the problem of structural reliability analysis is reduced to the well known standard interval arithmetic [1]. However, there may be some degree of our belief to each expert judgement whose value is determined by experience and competence of the expert. Therefore, it is necessary to take into account the available information about experts to obtain more credible assessments of the structural reliability.

It should be noted that uncertainty in parameters can be considered in a framework of hierarchical uncertainty models which are rather common in uncertainty theory. Different application examples of these models can be found in [9, 11]. A comprehensive review of hierarchical models is given in [8] where it is argued that the most common hierarchical model is the Bayesian one [5, 13, 14, 15, 36]. At the same time, the Bayesian hierarchical model is unrealistic in problems where there is available only partial information about the system behavior.

Therefore, models studied in this paper can be regarded as an extension of the Bayesian hierarchical model to the case of imprecise parameters of probability distributions. Numerical examples illustrate the proposed approach.

The exhaustive review and analysis of expert's elicitation procedures are given in [2, 7]. Therefore, these questions are remained outside the paper. It is supposed here that assessments of parameters of probability distributions in the form of their intervals are available and each assessment is characterized by some probability (belief) or interval-valued probability that a true value of the assessed parameter is in a given interval. This implies that the term "expert information" is used in the paper in a more general sense. For example, confidence intervals of parameters elicited as a result of statistical inference with corresponding confidence probabilities can also be considered as the "expert information".

2 Preliminary definitions

Suppose the continuous random variable $X(x)$ is defined on the sample space Ω and information about this variable is represented as a set of m interval-valued expectations of functions $f_1(X),...,f_m(X)$. Denote these lower and upper expectations $\underline{\mathbb{E}}_\pi f_i$ and $\overline{\mathbb{E}}_\pi f_i$, $i=1,...,m$. In terms of the theory of imprecise probabilities the corresponding functions $f_i(X)$ and interval-valued expectations $\underline{\mathbb{E}}_\pi f_i$ and $\overline{\mathbb{E}}_\pi f_i$ are called *gambles* and *lower and upper previsions*, respectively. Various types of information can be modelled by means of lower and upper previsions. For example, if f_i is the indicator function of an event A, then previsions $\underline{\mathbb{E}}_\pi f_i$ and $\overline{\mathbb{E}}_\pi f_i$ can be regarded as lower and upper probabilities of the event A. If $f_i(X)=X$, then $\underline{\mathbb{E}}_\pi f_i$ and $\overline{\mathbb{E}}_\pi f_i$ are bounds for a mean value of the corresponding random variable. The lower and upper previsions $\underline{\mathbb{E}}_\pi f_i$ and $\overline{\mathbb{E}}_\pi f_i$ can also be interpreted as bounds for an unknown precise prevision $\mathbb{E}_\pi f_i$ which will be called by a *linear prevision*.

For computing new previsions $\underline{\mathbb{E}}_\pi g$ and $\overline{\mathbb{E}}_\pi g$ of a gamble $g(X)$ from the available information, *natural extension* can be written as the following optimization problems:

$$\underline{\mathbb{E}}_\pi g=\min_\pi \int_\Omega g(x)\pi(x)\mathrm{d}x,\ \overline{\mathbb{E}}_\pi g=\max_\pi \int_\Omega g(x)\pi(x)\mathrm{d}x, \tag{1}$$

subject to

$$\pi(x)\geq 0,\ \int_\Omega \pi(x)\mathrm{d}x=1,\ \underline{\mathbb{E}}_\pi f_i\leq \int_\Omega f_i(x)\pi(x)\mathrm{d}x\leq \overline{\mathbb{E}}_\pi f_i,\ i\leq m. \tag{2}$$

Here the minimum and maximum are taken over a set of all possible probability density functions $\{\pi(x)\}$ satisfying conditions (2).

It should be noted that optimization problems (1)-(2) are linear and dual optimization problems can be written as follows [17, 19, 30]:

$$\overline{\mathbb{E}}_\pi g=\min_{c_0,c_i,d_i}\left(c_0+\sum_{i=1}^m\left(c_i\overline{\mathbb{E}}_\pi f_i-d_i\underline{\mathbb{E}}_\pi f_i\right)\right), \tag{3}$$

$$\underline{\mathbb{E}}_\pi g=-\overline{\mathbb{E}}_\pi(-g), \tag{4}$$

subject to c_i, $d_i\in\mathbb{R}_+$, $c_0\in\mathbb{R}$, $i=1,...,m$, and $\forall x\in\Omega$,

$$c_0+\sum_{i=1}^m\left(c_i-d_i\right)f_i(x)\geq g(x). \tag{5}$$

Here c_0, c_i, d_i are optimization variables such that c_0 corresponds to the constraint $\int_\Omega \pi(x)\mathrm{d}x = 1$ in (15), c_i corresponds to $\int_\Omega f_i(x)\pi(x)\mathrm{d}x \leq \overline{\alpha}_i$, d_i corresponds to $\underline{\alpha}_i \leq \int_\Omega f_i(x)\pi(x)\mathrm{d}x$.

Optimization problems (3)-(5) are no linear programming problems in the usual sense because they contain an infinite number of constraints (5) which are defined for each value x from Ω. This implies that these problems can generally be solved only approximately or analytically in some special cases [17]. However, it will be shown below that if all functions f_i are indicator functions of some events (corresponding previsions are probabilities), then the infinite number of constraints is always reduced to a finite number.

In order to indicate that expectations are taken with respect to the density π, we will denote them $\mathbb{E}_\pi$.

3 Imprecise stress-strength model

Let us briefly consider an approach to the structural reliability computation proposed in [32, 31]. Suppose that available information about the stress and strength is given as a set of n lower $\underline{\mathbb{E}}_\pi h_i$ and upper $\overline{\mathbb{E}}_\pi h_i$ previsions of gambles $h_i(X,Y)$ such that

$$\underline{\mathbb{E}}_\pi h_i \leq \int_{\mathbb{R}_+^2} h_i(x,y)\pi(x,y)\mathrm{d}x\mathrm{d}y \leq \overline{\mathbb{E}}_\pi h_i,\ i \leq n.$$

It is not excluded that the source evidence can be precisely known and $\underline{\mathbb{E}}_\pi h_i = \overline{\mathbb{E}}_\pi h_i$.

Taking into account that $R = \Pr\{X \leq Y\} = \Pr\{0 \leq Y - X \leq \infty\}$, we can write the following optimization problems for computing $\underline{R}$ and $\overline{R}$ given the source partial knowledge is a set of interval-valued statistical characteristics:

$$\underline{R} = \inf_\pi \int_{\mathbb{R}_+^2} I_{[0,\infty)}(y-x)\pi(x,y)\mathrm{d}x\mathrm{d}y,\ \overline{R} = \sup_\pi \int_{\mathbb{R}_+^2} I_{[0,\infty)}(y-x)\pi(x,y)\mathrm{d}x\mathrm{d}y, \tag{6}$$

subject to

$$\pi(x,y) \geq 0,\ \int_{\mathbb{R}_+^2} \pi(x,y)\mathrm{d}x\mathrm{d}y = 1,\ \underline{\mathbb{E}}_\pi h_i \leq \int_{\mathbb{R}_+^2} h_i(x,y)\pi(x,y)\mathrm{d}x\mathrm{d}y \leq \overline{\mathbb{E}}_\pi h_i,\ i \leq n. \tag{7}$$

Here the infimum and supremum are taken over the set of all possible probability density functions $\{\pi(x,y)\}$ satisfying conditions (7) and $I_{[0,\infty)}(y-x)$ is the indicator function such that $I_{[0,\infty)}(y-x) = 1$ if $y - x \geq 0$ and $I_{[0,\infty)}(y-x) = 0$ if $y - x < 0$. That is, no assumptions about specific probability distributions is introduced. It is also worth noticing that problems (7) can be solved without having to introduce an assumption about the independence of variables X and Y. The analyst may be ignorant of whether the variables are dependent or not.

If random variables X and Y are independent, the additional constraint $\pi(x,y) = \pi_X(x)\pi_Y(y)$ is added to constraints (7), where π_X and π_Y are densities of X and Y.

Example 1. Suppose the following partial information is available:

1. the mean value of the stress is known precisely 8 and the second moment is interval-valued $[40, 70]$;
2. the probability of finding the strength in the interval $[0, 12]$ is also known precisely, let us say 0.1.

It is assumed that the stress and strength are statistically independent and the objective is to find structural reliability based on the above scarce information. Now one can state the corresponding optimization problems

$$\underline{R} = \inf_{\pi_X,\pi_Y} \int_{\mathbb{R}_+^2} I_{[0,\infty)}(y-x)\pi_X(x)\pi_Y(y)\mathrm{d}x\mathrm{d}y,\ \overline{R} = \sup_{\pi_X,\pi_Y} \int_{\mathbb{R}_+^2} I_{[0,\infty)}(y-x)\pi_X(x)\pi_Y(y)\mathrm{d}x\mathrm{d}y,$$

subject to

$$\int_{\mathbb{R}_+} \pi_X(x)\mathrm{d}x = 1,\ \pi_X(x) \geq 0,\ \int_{\mathbb{R}_+} \pi_Y(x)\mathrm{d}x = 1,\ \pi_X(x) \geq 0,$$

$$\int_{\mathbb{R}_+} x\pi_X(x)\mathrm{d}x = 8,\ 40 \leq \int_{\mathbb{R}_+} x^2\pi_X(x)\mathrm{d}x \leq 70,\ \int_{\mathbb{R}_+} I_{[0,12]}(y)\pi_Y(y)\mathrm{d}y = 0.1.$$

By solving the above problems numerically, we obtain $\underline{R} = 0.7$ and $\overline{R} = 1$.

4 The problem statement

Suppose that the stress and strength are governed by the probability density functions $\pi_X(x|\Theta_X)$ and $\pi_Y(y|\Theta_Y)$ or by the cumulative distribution functions $F_X(x|\Theta_X)$ and $F_Y(y|\Theta_Y)$, respectively, where $\Theta_X = (\theta_1, ..., \theta_k)$ and $\Theta_Y = (\vartheta_1, ..., \vartheta_n)$ are vectors of parameters. By assuming the vectors Θ_X and Θ_Y are continuous random variables defined on the sample spaces Λ_X and Λ_Y, respectively, information about parameters can be represented in the following form:

$$\underline{\alpha}_j \leq \mathbb{E}_\rho f_j(\Theta_X) \leq \overline{\alpha}_j,\ j = 1, ..., m,\ \underline{\beta}_j \leq \mathbb{E}_\omega g_j(\Theta_Y) \leq \overline{\beta}_j,\ j = 1, ..., l, \tag{8}$$

or

$$\underline{\alpha}_j \leq \int_{\Lambda_X} f_j(\Theta_X)\rho(\Theta_X)\mathrm{d}\Theta_X \leq \overline{\alpha}_j,\ j \leq m,\ \underline{\beta}_j \leq \int_{\Lambda_Y} g_j(\Theta_Y)\omega(\Theta_Y)\mathrm{d}\Theta_Y \leq \overline{\beta}_j,\ j \leq l. \tag{9}$$

Here ρ and ω are unknown joint density functions of vectors Θ_X and Θ_Y satisfying constraints (8)-(9). This means that we know lower and upper expectations of some functions of parameters Θ_X and Θ_Y, such that m judgements are available about parameters of the distribution corresponding to the stress and l judgements are available about parameters of the distribution corresponding to the strength. In particular, if $\underline{\alpha}_j$ and $\overline{\alpha}_j$ are lower and upper probabilities that the j-th parameter is in the bounds a, b, then $f_j(\Theta_X)$ is the indicator function $I_{[a,b]}(\theta_j)$ such that $I_{[a,b]}(\theta_j) = 1$ if $\theta_j \in [a, b]$ and $I_{[a,b]}(\theta_j) = 0$ if $\theta_j \notin [a, b]$. By formalizing the information about parameters in the form of (8)-(9), we assume that parameters are random variables having the densities ρ and ω. In fact, there is a set of densities satisfying (8)-(9) and each density from this set can be regarded as a candidate to further analysis. If parameters are statistically independent as random variables, then the joint densities are represented as a product of marginal ones and this condition can be considered as some additional information about parameters.

The structural reliability can be described by the following probability:

$$R = \Pr\{X \leq Y\} = \mathbb{E}_\pi I_{[0,\infty)}(Y - X) = \int_{\mathbb{R}^2_+} I_{[0,\infty)}(y - x)\pi(x, y|\Theta)\mathrm{d}x\mathrm{d}y. \tag{10}$$

Here $\pi(x, y|\Theta)$ is a joint density of the stress and strength depending on a vector of parameters $\Theta = \Theta_X \cup \Theta_Y$. In particular, if the stress and strength are statistically independent, then there holds [6]

$$R = \int_{\mathbb{R}^2_+} I_{[0,\infty)}(y - x)\pi_X(x|\Theta_X)\pi_Y(y|\Theta_Y)\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}_+} \pi_Y(z|\Theta_Y)F_X(z|\Theta_X)\mathrm{d}z. \tag{11}$$

If there is no information about independence of the stress and strength, then, according to [32, 31], only bounds for the structural reliability can be found as

$$\underline{R} = \max_{z\geq 0} \max\{0, F_X(z|\Theta_X) - F_Y(z|\Theta_Y)\},\ \overline{R} = 1 - \max_{z\geq 0} \max\{0, F_Y(z|\Theta_Y) - F_X(z|\Theta_X)\}. \tag{12}$$

Our task is to find the system reliability measures taking into account imprecision of the parameters Θ_X and Θ_Y. The imprecision of parameters leads to imprecision of the stress-strength model. Therefore, we will find out the lower and upper bounds for the reliability.

In order to give readers the essence of the subject analyzed and make all the formulas more readable, we will assume for simplicity that $k = 1$ and $n = 1$, i.e., $\Theta_X = (\theta)$ and $\Theta_Y = (\vartheta)$. Furthermore, throughout the paper obvious constraints for densities ρ (or π, ω) to optimization problems such that $\rho(x) \geq 0$, $\int_{\mathbb{R}_+} \rho(x)\mathrm{d}x = 1$ will not be written.

Four cases, concerning different conditions of independence of the stress and strength and their parameters, have to be studied:

1. independent stress and strength and independent parameters;
2. independent stress and strength and the lack of information about independence of parameters;
3. the lack of information about independence of the stress and strength and independent parameters;
4. the lack of information about independence of the stress, strength, and their parameters.

5 Independent stress and strength

Let us study a case when the stress and strength are statistically independent.

5.1 Independent parameters

Let us consider an event $B = \{X \leq Y\}$. By having the density functions $\pi_X(x|\theta)$ and $\pi_Y(y|\vartheta)$, we can find the conditional probability $P(B|\theta,\vartheta)$ of the event B under condition that parameters of densities π_X and π_Y are θ and ϑ

$$P(B|\theta,\vartheta) = \int_{\mathbb{R}_+^2} I_B(x,y)\pi_X(x|\theta)\pi_Y(y|\vartheta)\mathrm{d}x\mathrm{d}y.$$

At the same time, by assuming that the parameters are random variables defined on the sample spaces Λ_X and Λ_Y and having densities ρ and ω, one can state

$$R = P(B) = \int_{\Lambda_X}\int_{\Lambda_Y} P(B|\theta,\vartheta)\rho(\theta)\omega(\vartheta)\mathrm{d}\theta\mathrm{d}\vartheta.$$

This corresponds to the Bayesian second-order uncertainty model which can always be reduced to a first-order model, by "integrating out the higher-order parameters“ [5, 13, 14, 15, 36].

Since the densities ρ and ω are unknown and there are available only constraints (8)-(9) for a set of all possible densities, then we find only lower and upper bounds for $P(B)$ by solving the following optimization problems:

$$\underline{R} = \min_{\rho,\omega}\int_{\Lambda_X}\int_{\Lambda_Y} G(\theta,\vartheta)\rho(\theta)\omega(\vartheta)\mathrm{d}\theta\mathrm{d}\vartheta,\ \overline{R} = \max_{\rho,\omega}\int_{\Lambda_X}\int_{\Lambda_Y} G(\theta,\vartheta)\rho(\theta)\omega(\vartheta)\mathrm{d}\theta\mathrm{d}\vartheta, \tag{13}$$

subject to (8)-(9).

Here

$$G(\theta,\vartheta) = P(B|\theta,\vartheta) = \int_{\mathbb{R}_+^2} I_B(x,y)\pi_X(x|\theta)\pi_Y(y|\vartheta)\mathrm{d}x\mathrm{d}y,$$

and the minimum and maximum are taken over all possible densities ρ and ω satisfying (8)-(9).

The obtained non-linear optimization problems can be numerically solved by means of the following approximate algorithm.

Algorithm for computing the structural reliability.

1. Let us restrict sets of all values of θ and ϑ by some intervals $[\theta_{\min},\theta_{\max}]$ and $[\vartheta_{\min},\vartheta_{\max}]$ and divide these intervals into N_1 and N_2 subintervals. As a result, we obtain N_1+1 points $(\theta_1,...,\theta_{N_1+1})$ and N_2+1 points $(\vartheta_1,...,\vartheta_{N_2+1})$. Here $\theta_{\min} = \theta_1$, $\theta_{\max} = \theta_{N_1+1}$, $\vartheta_{\min} = \vartheta_1$, $\vartheta_{\max} = \vartheta_{N_2+1}$. Now the optimization problem for computing the lower reliability can be rewritten as follows:

$$\underline{R} = \min_{\rho_i',\omega_k'}\sum_{i=1}^{N_1+1}\left(\sum_{k=1}^{N_2+1} G(\theta_i,\vartheta_k)\omega_k'\right)\rho_i',$$

   subject to

$$\underline{\alpha}_j \leq \sum_{i=1}^{N_1+1} f_j(\theta_i)\rho_i' \leq \overline{\alpha}_j,\ j=1,...,m,\ \underline{\beta}_j \leq \sum_{k=1}^{N_2+1} g_j(\vartheta_k)\omega_k' \leq \overline{\beta}_j,\ j=1,...,l,$$

$$\sum_{i=1}^{N_1+1}\rho_i' = 1,\ \sum_{k=1}^{N_2+1}\omega_k' = 1.$$

   Here $\rho_i' = (\theta_{i+1}-\theta_i)\rho(\theta_i)$ and $\omega_k' = (\vartheta_{k+1}-\vartheta_k)\omega(\vartheta_k)$. But we do not need to obtain optimal densities ρ and ω because $\underline{R}$ depends only on ρ_i' and ω_k'. Therefore, we use ρ_i' and ω_k' as variables of optimization without considering their original sense.

2. Since constraints for ω_k' and for ρ_i' are separated and the minimum of the objective function is achieved by minimal terms in brackets (due to the condition $\rho_i' \geq 0$), then, by fixing the value $\theta = \theta_i$, we can find the values $\underline{R}(\theta_i)$, $i=1,...,N_1+1$, as solutions to N_1+1 linear programming problems

$$\underline{R}(\theta_i) = \min_{\omega_k'}\sum_{k=1}^{N_2+1} G(\theta_i,\vartheta_k)\omega_k',$$

subject to

$$\underline{\beta}_j \leq \sum_{k=1}^{N_2+1} g_j(\vartheta_k)\omega_k' \leq \overline{\beta}_j,\ j=1,...,l,\ \sum_{k=1}^{N_2+1} \omega_k' = 1.$$

3. Now we obtain the following linear optimization problem:

$$\underline{R} = \min_{\rho_i'} \sum_{i=1}^{N_1+1} \underline{R}(\theta_i)\rho_i',$$

subject to

$$\underline{\alpha}_j \leq \sum_{i=1}^{N_1+1} f_j(\theta_i)\rho_i' \leq \overline{\alpha}_j,\ j=1,...,m,\ \sum_{i=1}^{N_1+1} \rho_i' = 1.$$

So, it is necessary to solve N_1+2 linear programming problems having N_2+1 variables. They can be solved by means of the well known simplex methods. The problem for computing the upper bound is similarly solved. The accuracy of solutions is defined by values of N_1 and N_2. However, if all functions f_j and g_j are indicator functions, i.e., the available information about the stress and strength is restricted by probabilities, then the proposed algorithm provides an exact solution.

It should be noted that the proposed algorithm can be applied to solve non-linear optimization problems (13) if constraints for ω and for ρ are separated and there are available lower and upper previsions of gambles $f_j(\theta)$ and $g_j(\vartheta)$. Generally, the algorithm produces an approximate interval $[\underline{R},\overline{R}]$ corresponding to so called free product [19] as a kind of independence. If we have some information represented as lower and upper previsions of a gamble $h(\theta,\vartheta)$ depending simultaneously on both random variables θ and ϑ, then the algorithm can not be used. This is its main limitation.

Let us consider a special case when the stress and strength are governed by exponential probability distributions with parameters μ and λ. Then $\theta=\mu$, $\vartheta=\lambda$, $\Lambda_X=\mathbb{R}_+$, $\Lambda_Y=\mathbb{R}_+$,

$$\pi_X(x|\theta) = \mu\exp(-\mu x),\ \pi_Y(y|\vartheta) = \lambda\exp(-\lambda y),$$
$$F_X(x|\theta) = 1-\exp(-\mu x),\ F_Y(y|\vartheta) = 1-\exp(-\lambda x).$$

Hence

$$G(\mu,\lambda) = \int_{\mathbb{R}_+} \lambda\exp(-\lambda y)(1-\exp(-\mu x))\mathrm{d}z = \mu/(\mu+\lambda).$$

This implies that

$$\underline{R} = \min_{\rho,\omega} \int_{\mathbb{R}_+^2} \frac{\mu}{\mu+\lambda}\rho(\mu)\omega(\lambda)\mathrm{d}\mu\mathrm{d}\lambda,\ \overline{R} = \max_{\rho,\omega} \int_{\mathbb{R}_+^2} \frac{\mu}{\mu+\lambda}\rho(\mu)\omega(\lambda)\mathrm{d}\mu\mathrm{d}\lambda,$$

subject to

$$\underline{\alpha}_j \leq \int_{\mathbb{R}_+} f_j(\mu)\rho(\mu)\mathrm{d}\mu \leq \overline{\alpha}_j,\ j=1,...,m,\ \underline{\beta}_j \leq \int_{\mathbb{R}_+} g_j(\lambda)\omega(\lambda)\mathrm{d}\lambda \leq \overline{\beta}_j,\ j=1,...,l.$$

Example 2. Suppose that the stress and strength are governed by exponential distributions. Two experts provide the following information about the parameter λ: the first expert - $\lambda\in[2,3]$ and the second expert - $\lambda\in[2,5]$. The belief to the first expert is 0.9. This means that the expert provides 90% of true judgements. The belief to the second expert is between 0.3 and 1. This means that the expert provides greater than 30% of true judgements. One expert provides the following information about the parameter μ: $\mu\in[8,10]$. The belief to the expert is 0.8. In order to find the structural reliability, it is necessary to solve the following optimization problems:

$$\underline{R} = \min_{\rho,\omega} \int_{\mathbb{R}_+^2} \frac{\mu}{\mu+\lambda}\rho(\mu)\omega(\lambda)\mathrm{d}\mu\mathrm{d}\lambda,\ \overline{R} = \max_{\rho,\omega} \int_{\mathbb{R}_+^2} \frac{\mu}{\mu+\lambda}\rho(\mu)\omega(\lambda)\mathrm{d}\mu\mathrm{d}\lambda,$$

subject to

$$0.5 \leq \int_{\mathbb{R}_+} I_{[2,3]}(\lambda)\omega(\lambda)\mathrm{d}\lambda \leq 0.5,\ 0.3 \leq \int_{\mathbb{R}_+} I_{[2,5]}(\lambda)\omega(\lambda)\mathrm{d}\lambda \leq 1,$$
$$0.8 \leq \int_{\mathbb{R}_+} I_{[8,10]}(\mu)\rho(\mu)\mathrm{d}\mu \leq 0.8.$$

Let us solve the above problems by means of the proposed algorithm.

1. Let us take 4 points $(10^{-20}, 8, 10, 10^{20})$ for μ and 5 points $(10^{-20}, 2, 3, 5, 10^{20})$ for λ, i.e.,
$$(\mu_1, ..., \mu_4) = (10^{-20}, 8, 10, 10^{20}),\ (\lambda_1, ..., \lambda_5) = (10^{-20}, 2, 3, 5, 10^{20}).$$

2. Now we have to find $\underline{R}(\mu_i),\ i = 1, ..., 4$. For example, we compute $\underline{R}(8)$ by solving the following linear programming problem
$$\underline{R}(8) = \min_{\omega'_k} \sum_{k=1}^{5} \frac{8}{8+\lambda_i}\omega'_k,$$
subject to
$$0.5 \leq \sum_{k=1}^{5} I_{[2,3]}(\lambda_k)\omega'_k \leq 0.5,\ 0.3 \leq \sum_{k=1}^{5} I_{[2,5]}(\lambda_k)\omega'_k \leq 1,\ \sum_{k=1}^{5} \omega'_k = 1.$$
This problem can be rewritten as
$$\underline{R}(8) = \min_{\omega'_k}(\omega'_1 + 0.8\omega'_2 + 0.73\omega'_3 + 0.62\omega'_4),$$
subject to
$$0.5 \leq \omega'_2 + \omega'_3 \leq 0.5,\ 0.3 \leq \omega'_2 + \omega'_3 + \omega'_4 \leq 1,\ \omega'_1 + \omega'_2 + \omega'_3 + \omega'_4 + \omega'_5 = 1.$$
Hence $\underline{R}(8) = 0.36$. Similarly, all values $\underline{R}(\mu_i)$ can be found $\underline{R}(0) = 0$, $\underline{R}(10) = 0.39$, $\underline{R}(10^{20}) = 0.75$.

3. Now we obtain the following linear optimization problem:
$$\underline{R} = \min_{\rho'_i}(0.36\rho'_2 + 0.39\rho'_3 + 0.75\rho'_4),$$
subject to
$$0.8 \leq \rho'_2 + \rho'_3 \leq 0.8,\ \rho'_1 + \rho'_2 + \rho'_3 + \rho'_4 = 1.$$
Hence $\underline{R} = 0.29$. The upper reliability can be similarly computed $\overline{R} = 0.93$. It should be noted that the obtained solutions are exact because constraints contain only indicator functions.

5.2 The lack of information about independence of parameters

Suppose that parameters of the stress and strength are elicited from experts. Experts often share a great deal of technical background, and the expert independence assumption is highly questionable. This implies that θ and ϑ may be dependent, but the degree of dependency is unknown. In this case, we can assert that there is no information about independence of θ and ϑ. Then the joint density $\rho(\theta, \vartheta)$ can not be represented as a product of the marginal densities $\rho(\theta)$ and $\omega(\vartheta)$. This implies that objective functions (13) are of the form:
$$\underline{R} = \min_{\rho} \int_{\Lambda_X}\int_{\Lambda_Y} G(\theta, \vartheta)\rho(\theta, \vartheta)\mathrm{d}\theta\mathrm{d}\vartheta,\ \overline{R} = \max_{\rho} \int_{\Lambda_X}\int_{\Lambda_Y} G(\theta, \vartheta)\rho(\theta, \vartheta)\mathrm{d}\theta\mathrm{d}\vartheta. \tag{14}$$
Note that
$$\int_{\Lambda_X} f_j(\theta)\rho(\theta)\mathrm{d}\theta = \int_{\Lambda_X}\int_{\Lambda_Y} f_j(\theta)\rho(\theta, \vartheta)\mathrm{d}\theta\mathrm{d}\vartheta,\ \int_{\Lambda_Y} g_j(\vartheta)\omega(\vartheta)\mathrm{d}\vartheta = \int_{\Lambda_X}\int_{\Lambda_Y} g_j(\vartheta)\rho(\theta, \vartheta)\mathrm{d}\theta\mathrm{d}\vartheta.$$
Then constraints (8)-(9) can be rewritten as follows:
$$\underline{\alpha}_j \leq \int_{\Lambda_X}\int_{\Lambda_Y} f_j(\theta)\rho(\theta, \vartheta)\mathrm{d}\theta\mathrm{d}\vartheta \leq \overline{\alpha}_j,\ \underline{\beta}_j \leq \int_{\Lambda_X}\int_{\Lambda_Y} g_j(\vartheta)\rho(\theta, \vartheta)\mathrm{d}\theta\mathrm{d}\vartheta \leq \overline{\beta}_j. \tag{15}$$
In fact, the constraints are not changed. We have changed only their writing. It should be noted that the above problems are linear programming problems with an infinite number of variables. Therefore, the dual linear programming problems can be written
$$\underline{R} = \max\left\{c_0 + \sum_{j=1}^{m}\left(c_j\underline{\alpha}_j - d_j\overline{\alpha}_j\right) + \sum_{j=1}^{l}\left(v_j\underline{\beta}_j - w_j\overline{\beta}_j\right)\right\}, \tag{16}$$

subject to $c_j, d_j, v_j, w_j \in \mathbb{R}_+$, $c_0 \in \mathbb{R}$, and $\forall\theta \in \Lambda_X$, $\forall\vartheta \in \Lambda_Y$,

$$c_0 + \sum_{j=1}^{m} (c_j - d_j)\, f_j(\theta) + \sum_{j=1}^{l} (v_j - w_j)\, g_j(\vartheta) \le G(\theta, \vartheta), \tag{17}$$

and

$$\overline{R} = \min \left\{ c_0 + \sum_{j=1}^{m} \left(c_j \overline{\alpha}_j - d_j \underline{\alpha}_j\right) + \sum_{j=1}^{l} \left(v_j \overline{\beta}_j - w_j \underline{\beta}_j\right) \right\}, \tag{18}$$

subject to $c_j, d_j, v_j, w_j \in \mathbb{R}_+$, $c_0 \in \mathbb{R}$, and $\forall\theta \in \Lambda_X$, $\forall\vartheta \in \Lambda_Y$,

$$c_0 + \sum_{j=1}^{m} (c_j - d_j)\, f_j(\theta) + \sum_{j=1}^{l} (v_j - w_j)\, g_j(\vartheta) \ge G(\theta, \vartheta). \tag{19}$$

It turns out that dual optimization problems in many applications are simpler in comparison with problems (14)-(15) because this representation allows us to avoid the situation when a number of optimization variables is infinite. Of course, dual optimization problems have generally an infinite number of constraints each of them is defined by values of θ and ϑ. However, the number of constraints can usually be reduced to a finite number. Moreover, dual problems have a certain sense. Lower and upper bounds for the structural reliability are computed as some kind of linear approximation defined by the set of available data about the stress and strength. The more information we have, the more precise assessments of reliability can be obtained if this initial information is not contradictory.

Let us consider a special case when the stress and strength are governed by exponential probability distributions with parameters μ and λ. Then constraints to the optimization problem for computing the lower reliability are

$$c_0 + \sum_{j=1}^{m} (c_j - d_j)\, f_j(\mu) + \sum_{j=1}^{l} (v_j - w_j)\, g_j(\lambda) \le \mu/(\mu + \lambda).$$

Constraints for the upper reliability are

$$c_0 + \sum_{j=1}^{m} (c_j - d_j)\, f_j(\mu) + \sum_{j=1}^{l} (v_j - w_j)\, g_j(\lambda) \ge \mu/(\mu + \lambda).$$

Example 3. Let us compute bounds for the structural reliability by using data described in Example 2 and assuming that there is no information about independence of the parameters λ and μ. The lower reliability is determined as

$$\underline{R} = \max\{c_0 + 0.8\,(c_1 - d_1) + 0.5(v_1 - w_1) + (0.3v_2 - w_2)\},$$

subject to $c_i, d_i, v_i, w_i \in \mathbb{R}_+$, $c_0 \in \mathbb{R}$, and $\forall\mu \ge 0$, $\forall\lambda \ge 0$,

$$c_0 + I_{[8,10]}(\mu)\,(c_1 - d_1) + I_{[2,3]}(\lambda)(v_1 - w_1) + I_{[2,5]}(\lambda)(v_2 - w_2) \le \mu/(\mu + \lambda).$$

Let us divide the set of values μ into three intervals $[0, 8]$, $[8, 10]$, $[10, \infty)$, and the set of values λ into four intervals $[0, 2]$, $[2, 3]$, $[3, 5]$, $[5, \infty)$. The indicator functions are not changed if λ and μ lie inside these intervals, respectively. This implies that an infinite number of constraints is reduced to 12 constraints with minimal right sides, i.e., with minimal values of G. The function G decreases as μ decreases in the interval $[0, 1)$ and λ increases in the same interval. Consequently, for each pair of intervals for λ and μ, we have to take the maximal value of λ and the minimal value of μ. Consider, for example, the intervals $[8, 10]$ and $[3, 5]$. They form the following constraint:

$$c_0 + (c_1 - d_1) + (v_2 - w_2) \le G(8, 5) = 0.615.$$

Similarly, we can write all constraints

$$\begin{aligned}
c_0 + (v_1 - w_1) + (v_2 - w_2) &\le 0, \\
c_0 + (v_2 - w_2) &\le 0, \\
c_0 + (c_1 - d_1) &\le 0, \\
c_0 + (c_1 - d_1) + (v_1 - w_1) + (v_2 - w_2) &\le 0.73, \\
c_0 + (c_1 - d_1) + (v_2 - w_2) &\le 0.615, \\
c_0 &\le 0.
\end{aligned}$$

It should be noted that the number of constraints is less than 12 because some constraints follow from the above constraints. Hence $\underline{R} = 0.22$. For computing the upper bound $\overline{R}$, it is necessary to solve the following optimization problem:

$$\overline{R} = \min\{c_0 + 0.8\,(c_1 - d_1) + 0.5(v_1 - w_1) + (v_2 - 0.3w_2)\},$$

subject to $c_i, d_i, v_i, w_i \in \mathbb{R}_+$, $c_0 \in \mathbb{R}$, and

$$\begin{aligned} c_0 + (v_1 - w_1) + (v_2 - w_2) &\geq 1, \\ c_0 + (v_2 - w_2) &\geq 1, \\ c_0 + (c_1 - d_1) &\geq 1, \\ c_0 + (c_1 - d_1) + (v_1 - w_1) + (v_2 - w_2) &\geq 0.83, \\ c_0 + (c_1 - d_1) + (v_2 - w_2) &\geq 0.77, \\ c_0 &\geq 1. \end{aligned}$$

Here for each pair of intervals for λ and μ, we took the minimal value of λ and the maximal value of μ. The solution is $\overline{R} = 0.95$.

6 The lack of information about independence of the stress and strength

The lack of information about independence of the stress and strength does not mean that the parameters θ and ϑ of distributions are independent or not. Therefore, we consider two cases: independent parameters and the lack of information about their independence.

6.1 Independent parameters

In this case, expressions (12) are applied. Objective functions coincide with (13), but the function $G(\theta, \vartheta)$ can be represented now only as an interval with the bounds $\underline{G}(\theta, \vartheta)$ and $\overline{G}(\theta, \vartheta)$ which are of the form:

$$\underline{G}(\theta, \vartheta) = \max_{z \geq 0} \max\{0, F_X(z|\theta) - F_Y(z|\vartheta)\}, \quad \overline{G}(\theta, \vartheta) = 1 - \max_{z \geq 0} \max\{0, F_Y(z|\vartheta) - F_X(z|\theta)\}.$$

So, the problem of computing lower and upper bounds for the structural reliability under condition of the lack of information about independence of the stress and strength is reduced to the case considered in Section 5, i.e.,

$$\underline{R} = \min_{\rho,\omega} \int_{\Lambda_X} \int_{\Lambda_Y} \underline{G}(\theta, \vartheta)\rho(\theta)\omega(\vartheta)\mathrm{d}\theta\mathrm{d}\vartheta, \quad \overline{R} = \max_{\rho,\omega} \int_{\Lambda_X} \int_{\Lambda_Y} \overline{G}(\theta, \vartheta)\rho(\theta)\omega(\vartheta)\mathrm{d}\theta\mathrm{d}\vartheta,$$

subject to (8)-(9).

Let us consider a special case when the stress and strength are governed by exponential probability distributions with parameters μ and λ. Note that the function

$$F_X(z|\mu) - F_Y(z|\lambda) = \exp(-\lambda z) - \exp(-\mu z)$$

in non-negative if $\lambda \leq \mu$. Moreover, the maximum of this function is achieved at

$$z_{opt} = \frac{\ln \mu - \ln \lambda}{\mu - \lambda}.$$

Then there holds

$$\underline{G}(\mu, \lambda) = \begin{cases} 0, & \lambda \geq \mu \\ e^{-\lambda z_{opt}} - e^{-\mu z_{opt}}, & \lambda < \mu \end{cases}.$$

Similarly, we can find $\overline{G}(\mu, \lambda)$

$$\overline{G}(\mu, \lambda) = 1 - \begin{cases} e^{-\mu z_{opt}} - e^{-\lambda z_{opt}}, & \lambda > \mu \\ 0, & \lambda \leq \mu \end{cases}.$$

Denote $\gamma = \lambda/\mu$ and $\eta = \mu/\lambda$ ($\gamma = 1/\eta$). Then there hold

$$\underline{G}(\mu, \lambda) = \begin{cases} 0, & \gamma \geq 1 \\ \gamma^{-\frac{\gamma}{\gamma-1}} - \gamma^{-\frac{1}{\gamma-1}}, & \gamma \in [0,1) \end{cases}, \quad \overline{G}(\mu, \lambda) = 1 - \begin{cases} \eta^{-\frac{\eta}{\eta-1}} - \eta^{-\frac{1}{\eta-1}}, & \eta \in [0,1) \\ 0, & \eta \geq 1 \end{cases}.$$

Example 4. Let us compute bounds for the structural reliability by using data described in Example 2 and assuming that there is no information about independence of the stress and strength. By using the proposed approximate algorithm (see Example 2), we obtain $\underline{R} = 0.14$, $\overline{R} = 1$.

6.2 The lack of information about independence of parameters

In this case, optimization problems for computing lower and upper bounds for the reliability differ from optimization problems (14)-(15) and (16)-(19) with the function G. In particular, the function G is replaced by $\underline{G}$ in optimization problems for computing the lower bound for the reliability, and G is replaced by $\overline{G}$ in optimization problems for computing the upper reliability.

Let us consider a special case when the stress and strength are governed by exponential probability distributions with parameters μ and λ. Then constraints to the optimization problem for computing the lower reliability are

$$c_0 + \sum_{j=1}^{m} (c_j - d_j)\, f_j(\mu) + \sum_{j=1}^{l} (v_j - w_j)\, g_j(\lambda) \leq \left\{ \begin{array}{cc} 0, & \gamma \geq 1 \\ \gamma^{-\frac{\gamma}{\gamma-1}} - \gamma^{-\frac{1}{\gamma-1}}, & \gamma \in [0,1) \end{array} \right. .$$

Constraints for the upper reliability are

$$c_0 + \sum_{j=1}^{m} (c_j - d_j)\, f_j(\mu) + \sum_{j=1}^{l} (v_j - w_j)\, g_j(\lambda) \geq 1 - \left\{ \begin{array}{cc} \eta^{-\frac{\eta}{\eta-1}} - \eta^{-\frac{1}{\eta-1}}, & \eta \in [0,1) \\ 0, & \eta \geq 1 \end{array} \right. .$$

Example 5. Let us compute bounds for the structural reliability by using data described in Example 2 and assuming that there is no information about independence of parameters of the stress and strength. The lower reliability is determined as

$$\underline{R} = \max\{c_0 + 0.8\,(c_1 - d_1) + 0.5(v_1 - w_1) + (0.3v_2 - w_2)\},$$

subject to $c_i, d_i, v_i, w_i \in \mathbb{R}_+$, $c_0 \in \mathbb{R}$, and $\forall \mu \geq 0$, $\forall \lambda \geq 0$,

$$c_0 + I_{[8,10]}(\mu)\,(c_1 - d_1) + I_{[2,3]}(\lambda)(v_1 - w_1) + I_{[2,5]}(\lambda)(v_2 - w_2) \leq \left\{ \begin{array}{cc} 0, & \gamma \geq 1 \\ \gamma^{-\frac{\gamma}{\gamma-1}} - \gamma^{-\frac{1}{\gamma-1}}, & \gamma \in [0,1) \end{array} \right. .$$

Let us divide the set of values μ into three intervals $[0,8]$, $[8,10]$, $[10,\infty)$ and the set of values λ into four intervals $[0,2]$, $[2,3]$, $[3,5]$, $[5,\infty)$ (see Example 3). The function $\underline{G}$ decreases as γ increases in the interval $[0,1)$. Consequently, for each pair of intervals for λ and μ, we have to take the maximal value of λ and the minimal value of μ. Consider, for example, intervals $[8,10]$ and $[3,5]$. They form the following constraint:

$$c_0 + (c_1 - d_1) + (v_2 - w_2) \leq \underline{G}(8,5) = 0.17.$$

Similarly, we can write all constraints

$$\begin{aligned} c_0 + (v_1 - w_1) + (v_2 - w_2) &\leq 0, \\ c_0 + (v_2 - w_2) &\leq 0, \\ c_0 + (c_1 - d_1) &\leq 0, \\ c_0 + (c_1 - d_1) + (v_1 - w_1) + (v_2 - w_2) &\leq 0.35, \\ c_0 + (c_1 - d_1) + (v_2 - w_2) &\leq 0.17, \\ c_0 &\leq 0. \end{aligned}$$

Hence $\underline{R} = 0.11$. The upper bound $\overline{R} = 1$ can be found in the same way.

7 Conclusion

The approach to computing the structural reliability by imprecise parameters of the stress and strength probability distributions has been proposed in the paper. This approach takes into account the heterogeneity of the available information about parameters of probability distributions and different conditions of independence. Generally, computation of bounds for the structural reliability is reduced to solution of a number of linear programming problems. If initial information about parameters is represented as a set of interval-valued probabilities, then it is possible to obtain exact bounds. Otherwise, we get approximate bounds.

Examples (2)-(5) show that imprecision of results is strongly dependent on the condition of independence of the stress and strength or their parameters. In particular, we have obtained the following intervals for the reliability: $[0.29, 0.93] \subset [0.22, 0.95] \subset [0.14, 1] \subset [0.11, 1]$, which correspond to the independent stress and strength and independent parameters, the independent stress and strength and the lack of information about independence of parameters, the lack of information about independence of the stress and strength and independent parameters, the lack of information about independence of the stress, strength, and their parameters. It can be seen from results of examples that independence of the stress and strength is the most crucial from the precision point of view.

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