Stress-Strength Reliability Models Under Incomplete Information

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A new approach to compute structural reliability is proposed. The novelty of the approach is that initial statistical information concerning the stress and strength of engineered structures is partial. Different numerous cases of state of knowledge about the stress and strength are analysed. A set of canonical analytical expressions for computing imprecise structural reliability has been obtained, and a few examples are presented. The reliability models developed are generalisations of conventional one.

Keywords: structural reliability, stress and strength, partial information, imprecise probabilities, optimisation problem, distributions on nested intervals.

1 Introduction

An engineered structure during its lifetime is subjected to stresses or/and actions. The stresses may cause a change of the condition or state of the structure from an undamaged or intact state to a state of deterioration, damage or failure. The boundary between damage states and failed states is referred to as the set of limit states. The reliability of a structure is concerned with how likely it is that the structure will reach a limit state and enter a state of failure (*Guidelines for Design of Wind Turbines* 2001).

In structural design, the reliability of a structural component is evaluated with respect to one or more failure modes. The structural component is described by a set of stochastic basic variable grouped into one vector, including, e.g., its strength, stiffness, geometry, and loading (Barlow & Proschan 1965). Each of this variables is stochastic in the sense that it - owing to natural variability and possible other uncertainties - may take on random realizations according to some probability distribution. The conventional approach suggests to consider a particular probability distribution for the stochastic variables despite data are often very scarce to support any specific distribution. A sensitivity analysis or the introduction of interval-valued parameters of probability distributions can demonstrate how uncertainty in input data can be propagated into output values. Nonetheless, usually they do not question the very distribution law. Analysing a few classes of probability distributions (robust statistical analysis) is a very laborious task and can rarely be performed in practice as the analyst is confronted limited resources. Even though a kind of robust analysis is performed, the results may be not credible as the true probability distribution may be omitted.

Bayesian approach, suggesting to model uncertainty in the parameters of probability distributions by prior probability laws, is another alternative. One of the principle points of disagreement among researchers regarding Bayesian approach is that prior distributions are assumed precise and the "dogma of precision" averts many as precision can hardly be attributed to subjective judgements which the prior distributions are. The "dogma of precision" can partly be eliminated by involving in analysis whole families of probability distributions, which is called Bayesian sensitivity analysis. Yet, this kind of analysis is a hardly affordable task in practice due to too much resources needed.

Studies on generalised uncertainty analysis in structural reliability assessments which are closely related to the subject described in this paper can be found in (Ayyub & Lai 1992, Lai & Ayyub 1994) and (Chao & Ayyub 1996). Yet, the approaches suggested deal with ambiguity and vagueness in failure, as we consider imprecision as a source of uncertainty. The last section "Probability distributions on nested intervals" illustrates how one can arrive at possibility distributions in the framework of the approach suggested.

Nevertherless, we consider this case as very rarely encountered in practice.

There is another way of coping with uncertainty and that is addressed in this paper. It is based on the assumption that only partial information about the statistical behaviour of random variables is available. Partial information is thought as any statistical evidence reducing the set of possible probability distributions of the random variable. Assumptions on specific types of distributions are not necessary unless there is a firm evidence to support the assumptions. The results of partial knowledge modelling are interval-valued probability characteristics of interest that are obtained with a properly stated optimisation problem often called natural extension.

A set of canonical analytical expressions for computing imprecise structural reliability has been obtained, which is presented in this paper along with several examples.

2 The approach in general

Briefly the problem addressed in this paper can be stated as follows.

Let Y represent a random variable describing the strength of a system and let X represent a random variable describing the stress or load placed on the system. System failure occurs when the stress on the system exceeds the strength of the system: $\Omega = \{(x \in X, y \in Y) : X \geq Y\}$. Here Ω is a region where combination of system parameters lead to an unacceptable or unsafe system response. Then the reliability of the system is determined as $R = \Pr\{X \leq Y\}$, and the unreliability is determined as $Q = \Pr\{X > Y\} = 1 - R$.

Suppose that available information about the stress and strength is given as a set of n lower $\underline{r}_i$ and upper $\overline{r}_i$ previsions of gambles $h_i(x,y)$ such that

$$\underline{r}_i \leq \int_{\mathbf{R}_+} \int_{\mathbf{R}_+} h_i(x,y)\rho(x,y)\mathrm{d}x\mathrm{d}y \leq \overline{r}_i,\ i = 1, ..., n.$$

For example, if $\underline{r}_i$ and $\overline{r}_i$ are lower and upper bounds for the mean value of the stress, then the gamble is $h_i(x,y) = x$. If $\underline{r}_i$ and $\overline{r}_i$ are the bounds for the probability of strength being in interval $[a,b]$, then there holds $h_i(x,y) = I_{[a,b]}(y)$. Here $I_{[a,b]}(y)$ is an indicator function such that $I_{[a,b]}(y) = 1$ if $y \in [a,b]$, and $I_{[a,b]}(y) = 0$ otherwise. If it is, for instance, known that the second moment of the stress is at least k times as probable as the second moment of the strength, then $h_i(x,y) = x^2 - ky^2$. It is not excluded that the source evidence can be precisely known, yet, this case is a particular one and a point value can be thought as a degenerate interval $\int_{\mathbf{R}_+}\int_{\mathbf{R}_+} h_i(x,y)\rho(x,y)\mathrm{d}x\mathrm{d}y = r_k$.

Taking into account that $R = \Pr\{X \leq Y\} = \Pr\{0 \leq Y - X \leq \infty\}$ we can write the following problems for computing $\underline{R}$ and $\overline{R}$ given the source partial knowledge is a set of interval-valued statistical characteristics:

$$\underline{R} = \inf_{\mathcal{P}} \int_{\mathbf{R}_+}\int_{\mathbf{R}_+} I_{[0,\infty)}(y-x)\rho(x,y)\mathrm{d}x\mathrm{d}y, \tag{1}$$

$$\overline{R} = \sup_{\mathcal{P}} \int_{\mathbf{R}_+}\int_{\mathbf{R}_+} I_{[0,\infty)}(y-x)\rho(x,y)\mathrm{d}x\mathrm{d}y, \tag{2}$$

subject to

$$\int_{\mathbf{R}_+}\int_{\mathbf{R}_+} \rho(x,y)\mathrm{d}x\mathrm{d}y = 1,\ \rho(x,y) \geq 0, \tag{3}$$

$$\underline{r}_i \leq \int_{\mathbf{R}_+}\int_{\mathbf{R}_+} h_i(x,y)\rho(x,y)\mathrm{d}x\mathrm{d}y \leq \overline{r}_i,\ i \leq n. \tag{4}$$

Here the infimum and supremum are taken over the set $\mathcal{P}$ of all possible probability density functions $\{\rho(x)\}$ satisfying conditions (3)-(4). That is, no assumptions about specific probability distributions is introduced. It is also worth noticing that problem (1)-(4) can be solved without having to introduce an assumption about the independence of variables X and Y. The analyst may be ignorant of whether the variables are dependent or not.

For the sake of simplicity it is assumed that interval-valued statistical evidence on the stress and strength is separate and the evidence does not bind the two variables in one function. In this case constraints (4) can be rewritten

$$\underline{r}_i \leq \int_{\mathbf{R}_+}\int_{\mathbf{R}_+} f_i(x)\rho(x,y)\mathrm{d}x\mathrm{d}y \leq \overline{r}_i,\ 1 \leq i \leq n_1, \tag{5}$$

$$\underline{r}_i \leq \int_{\mathbf{R}_+}\int_{\mathbf{R}_+} g_i(y)\rho(x,y)\mathrm{d}x\mathrm{d}y \leq \overline{r}_i,\ n_1 \leq i \leq n_2. \tag{6}$$

If random variables X and Y are independent, then $\rho(x,y)=\rho_X(x)\rho_Y(y)$ and constraints (3), (5) and (6) can be modelled as

$$\int_{\mathbf{R}_+}\rho_X(x)\mathrm{d}x=1,\ \rho_X(x)\geq 0,\ \int_{\mathbf{R}_+}\rho_Y(y)\mathrm{d}y=1,\ \rho_Y(y)\geq 0,$$

$$\underline{r}_i\leq\int_{\mathbf{R}_+}f_i(x)\rho_X(x)\mathrm{d}x\leq\overline{r}_i,\ 1\leq i\leq n_1,$$

$$\underline{r}_i\leq\int_{\mathbf{R}_+}g_i(y)\rho_Y(y)\mathrm{d}y\leq\overline{r}_i,\ n_1\leq i\leq n_2.$$

The known tool for solving linear and non-linear optimisation problems is duality principle which may significantly simplify finding an analytical solution. Optimisation problem in the form (1)-(4) is called the primal form (Utkin & Kozine 2001). In some cases dual optimisation problems do not exist and solutions can be found by means of numerical methods as it is done in the example below.

Example 1 *Suppose the following partial information is available:*

1. *the mean value of a stress is known precisely* $m_1=8$ *and the second moment is interval-valued* $m_2\in[40,70]$*;*

2. *the probability of finding the strength in the interval* $[0,12]$ *is also known precisely, let us say* $p=0.1$.

   It is assumed that the stress and strength are statistically independent and the objective is to find structural reliability based on the above scarce information.

   Let us write the source evidence as expected/mean values $m_1=M(x)$*,* $m_2=M(x^2)$ *and* $p=M(I_{[0,12]}(y))$*, where* $M(\bullet)$ *is the operator of expectation. Now one can state the corresponding optimization problems*

$$\underline{R}\left(\overline{R}\right)=\inf_{\mathcal{P}}\left(\sup_{\mathcal{P}}\right)\int_{\mathbf{R}_+}\int_{\mathbf{R}_+}I_{[0,\infty)}(y-x)\rho_X(x)\rho_Y(y)\mathrm{d}x\mathrm{d}y,$$

subject to

$$\int_{\mathbf{R}_+}\rho_X(x)\mathrm{d}x=1,\ \int_{\mathbf{R}_+}\rho_Y(y)\mathrm{d}y=1,\ \rho_X(x)\geq 0,\ \rho_Y(y)\geq 0,$$

$$\int_{\mathbf{R}_+}x\rho_X(x)\mathrm{d}x=8,\ 40\leq\int_{\mathbf{R}_+}x^2\rho_X(x)\mathrm{d}x\leq 70,$$

$$\int_{\mathbf{R}_+} I_{[0,12]}(y)\rho_Y(y)\mathrm{d}y = 0.1.$$

By solving the above problems numerically, we obtain $\underline{R} = 0.7$ *and* $\overline{R} = 1$.

3 Partially known probability distributions

3.1 Independent X and Y

Let us consider a case when one knows precisely the probabilities

$$\Pr\{X \leq \alpha_i\} = p_i, \ \Pr\{Y \leq \beta_j\} = q_j, \ i = 1, ..., n, \ j = 1, ..., m.$$

Here it is assumed that

$$\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_n \text{ and } \beta_1 \leq \beta_2 \leq ... \leq \beta_m,$$

$$p_1 \leq p_2 \leq ... \leq p_n \text{ and } q_1 \leq q_2 \leq ... \leq q_m.$$

The second assumption is obvious because p_i and q_j are the values of probability distributions. In other words, only n points of the probability distribution of X and m points of the probability distribution of Y are known.

Suppose variables X and Y are independent. Then, problems (1)-(4) can be rewritten

$$\underline{R}(\overline{R}) = \inf_{\mathcal{P}}(\sup_{\mathcal{P}}) \int_{\mathbf{R}_+}\int_{\mathbf{R}_+} I_{[0,\infty)}(y-x)\rho_X(x)\rho_Y(y)\mathrm{d}x\mathrm{d}y, \tag{7}$$

$$\int_{\mathbf{R}_+} \rho_X(x)\mathrm{d}x = 1, \ \rho_X(x) \geq 0, \ \int_{\mathbf{R}_+} \rho_Y(y)\mathrm{d}y = 1, \ \rho_Y(y) \geq 0, \tag{8}$$

$$\int_{\mathbf{R}_+} I_{[0,\alpha_i]}(x)\rho_X(x)\mathrm{d}x = p_i, \ \int_{\mathbf{R}_+} I_{[0,\beta_j]}(y)\rho_Y(y)\mathrm{d}y = q_j, \ i = 1, ..., n, \ j = 1, ..., m. \tag{9}$$

In (Utkin & Kozine 2001) it was proven that solutions for optimisation problems similar to (7)-(9) exist on degenerate distributions, and referring to this property the following optimisation problem, equivalent to the above, can be stated:

$$\underline{R}(\overline{R}) = \inf\,(\sup) \sum_{k=1}^{n+1}\sum_{j=1}^{m+1} I_{[0,\infty)}(y_j - x_k)c_k d_j, \tag{10}$$

subject to

$$\sum_{k=1}^{n+1} c_k = 1,\ \sum_{j=1}^{m+1} d_i = 1, \tag{11}$$

$$\sum_{i=1}^{n+1} I_{[0,\alpha_k]}(x_i)c_i = p_k,\ \sum_{i=1}^{m+1} I_{[0,\beta_j]}(y_i)d_i = q_j,\ k = 1, ..., n,\ j = 1, ..., m. \tag{12}$$

Here the infimum and supremum are taken over the set of variables $x_i,\ y_j,\ c_i,\ d_j \in \mathbf{R}_+,\ i = 1, ..., n,$ $j = 1, ..., m$, subject to the constraints. These problems are non-linear either, which makes it usually difficult to obtain explicit analytical solutions. Nevertheless, for this particular case it is possible to do so and the theorem below gives us the solutions for problems (7)-(9) by employing the equivalent form (10)-(12).

Without loss in generality, it is assumed $p_0 = 0$, $q_0 = 0$, $p_{n+1} = 1$, $q_{m+1} = 1, \alpha_0 = 0$, $\beta_0 = 0$, $\alpha_{n+1} = A$ and $\beta_{m+1} = B$. In particular, one can assume $A \to \infty$ and $B \to \infty$. Let $j(i)$ be the minimal number j such that $\alpha_i \le \beta_{j(i)}$ and $l(k)$ be the minimal number l such that $\beta_k \le \alpha_{l(k)}$, i.e. $j(i) = \min\{j : \alpha_i \le \beta_j\}$, $l(k) = \min\{l : \beta_k \le \alpha_l\}$.

Theorem 1 *If stress X and strength Y are statistically independent and governed by partially known probability distributions in the form* $\Pr\{X \le \alpha_i\} = p_i,\ \Pr\{Y \le \beta_j\} = q_j,\ i = 1, ..., n,\ j = 1, ..., m,$ $\alpha_1 \le \alpha_2 \le ... \le \alpha_n$ *and* $\beta_1 \le \beta_2 \le ... \le \beta_m$, *then the lower and upper bounds for structural reliability are computed as follows:*

$$\underline{R} = \sum_{i=1}^{n}(p_i - p_{i-1})(1 - q_{j(i)}),\ j(i) = \min\{j : \alpha_i \le \beta_j\},\ \textit{and} \tag{13}$$

$$\overline{R} = 1 - \sum_{k=1}^{m}(q_k - q_{k-1})(1 - p_{l(k)}),\ l(k) = \min\{l : \beta_k \le \alpha_l\}. \tag{14}$$

Proof. Assume that

$$x_1 \le x_2 \le ... \le x_{n+1},\ y_1 \le y_2 \le ... \le y_{m+1}$$

are the values delivering inf and sup to the reliability computed according to objective function (10). Let us prove first that for all possible k values x_k and y_k, delivering the optima to this objective function, meet the following conditions: $x_k \in [\alpha_{k-1}, \alpha_k]$ and $y_k \in [\beta_{k-1}, \beta_k]$. Suppose that there are two optimal values of x_j and x_k such that $x_j \in [\alpha_{k-1}, \alpha_k]$ and $x_k \in [\alpha_{k-1}, \alpha_k]$. If $j < k$, then it follows from (12) that

$$c_1 + ... + c_j = p_k, \ c_1 + ... + c_j + c_{j+1} = p_k,$$

which is a contradiction.

If $j > k$, then it follows from (12) that

$$c_1 + ... + c_k = p_k, \ c_1 + ... + c_k + c_{k+1} = p_k,$$

which is also a contradiction.

Similarly, we arrive at contradictions for arbitrary number of values x_k belonging to the same interval. This implies that $x_k \in [\alpha_{k-1}, \alpha_k]$.

The proof of the condition $y_k \in [\beta_{k-1}, \beta_k]$ can be obtained in a similar way.

It follows from these conditions ($x_k \in [\alpha_{k-1}, \alpha_k]$ and $y_k \in [\beta_{k-1}, \beta_k]$) and from (12) that

$$c_1 = p_1, \ c_1 + c_2 = p_2, ..., \sum_{i=1}^{n} c_i = p_n,$$

$$d_1 = q_1, \ d_1 + d_2 = q_2, ..., \sum_{i=1}^{m} d_i = q_m.$$

Hence

$$c_k = p_k - p_{k-1}, \ d_j = q_j - q_{j-1}, \ k = 1, ..., n, \ j = 1, ..., m.$$

Note that the objective function (10) achieves its minimum if for all $k \leq n+1$ and $j \leq m+1$ there hold $I_{[0,\infty)}(y_j - x_k) = 0$. However, there exist values j and k such that $I_{[0,\infty)}(y_j - x_k) = 1$ for some combinations of y_j and x_k.

Let $j(k)$ be a minimal number j such that there hold $x_k \in [\alpha_{k-1}, \alpha_k]$, $y_{j(k)} \in [\beta_{j(k)-1}, \beta_{j(k)}]$ and $\alpha_k \leq \beta_{j(k)}$. Then $I_{[0,\infty)}(y_j - x_k) = 1$ for all $j \geq j(k) + 1$. Thus, it can be concluded

$$\underline{R} = \sum_{k=1}^{n+1} \sum_{j=j(k)+1}^{m+1} c_k d_j = \sum_{k=1}^{n+1} \sum_{j=j(k)+1}^{m+1} (p_k - p_{k-1})(q_j - q_{j-1}).$$

Taking into account that $\sum_{j=j(k)+1}^{m+1}(q_j - q_{j-1}) = 1 - q_{j(k)}$, the last formula is reduced to

$$\underline{R} = \sum_{k=1}^{n}(p_k - p_{k-1})(1 - q_{j(k)}).$$

For computing the upper bound $\overline{R}$, one can write

$$\overline{R} = \overline{P}(X < Y) = 1 - \underline{P}(X \geq Y) = 1 - \underline{Q}$$

$$= 1 - \inf \sum_{k=1}^{n+1}\sum_{j=1}^{m+1} I_{[0,\infty)}(x_k - y_j)c_k d_j.$$

Value $\underline{Q}$ can be obtained in a similar way as it was done for $\underline{R}$.

Let $l(k)$ be the minimal number such that there hold $x_{l(k)} \in [\alpha_{l(k)-1}, \alpha_{l(k)}]$, $y_k \in [\beta_{k-1}, \beta_k]$ and $\alpha_{l(k)} \geq \beta_k$. Then $I_{[0,\infty)}(x_l - y_k) = 1$ for all $l \geq l(k) + 1$. This implies that

$$\overline{R} = 1 - \underline{Q} = 1 - \sum_{k=1}^{m+1}\sum_{l=l(k)+1}^{n+1} c_l d_k = 1 - \sum_{k=1}^{m+1}\sum_{l=l(k)+1}^{n+1}(p_l - p_{l-1})(q_k - q_{k-1})$$

$$= 1 - \sum_{k=1}^{m}(q_k - q_{k-1})(1 - p_{l(k)}).$$

This formula completes the proof. ■

Corollary 1 *If X and Y are statistically independent and their probability distributions are known precisely, then*

$$\underline{R} = \overline{R} = \int_{\mathbf{R}_+} \rho_X(x)\,(1 - F_Y(x))\,\mathrm{d}x = 1 - \int_{\mathbf{R}_+} \rho_Y(x)\,(1 - F_X(x))\,\mathrm{d}x.$$

Here $F_X(x)$ and $F_Y(x)$ are the cumulative distribution functions of random variables X and Y, correspondingly, i.e. $F_X(x) = \Pr\{X \leq x\}$ and $F_Y(x) = \Pr\{Y \leq x\}$.

Proof. It follows from Theorem 1 that

$$\underline{R} = \lim_{n\to\infty}\sum_{i=1}^{n}(p_i - p_{i-1})(1 - q_{j(i)}) = \lim_{n\to\infty}\sum_{i=1}^{n}\rho_X(x_i)\triangle x_i(1 - F_Y(x_i))$$

$$= \int_{\mathbf{R}_+} \rho_X(x)\,(1 - F_Y(x))\,\mathrm{d}x.$$

$$\overline{R} = 1 - \lim_{m\to\infty}\sum_{k=1}^{m}\rho_Y(x_k)\triangle x_k(1 - F_X(x_k)) = 1 - \int_{\mathbf{R}_+} \rho_Y(x)\,(1 - F_X(x))\,\mathrm{d}x$$

$$= 1 - \int_{\mathbf{R}_+} \rho_Y(x) \left[\int_x^{\infty} \rho_X(y) \mathrm{d}y \right] \mathrm{d}x.$$

It is known (see, for example, (Dhillon 1999)) that $\int_{\mathbf{R}_+} \rho_X(x) (1 - F_Y(x)) \, \mathrm{d}x = 1 - \int_{\mathbf{R}_+} \rho_Y(x) \left[\int_x^{\infty} \rho_X(y) \mathrm{d}y \right] \mathrm{d}x$, which completes the proof. ■

Corollary 1 states that the obtained expressions coincide with conventional ones known in reliability theory, and this, in fact, means that Theorem 1 generalises the conventional formulas for structural reliability to interval-valued probabilities.

Now it is assumed that the probability distributions are more imprecise compared to the case for which Theorem 1 was proven.

Corollary 2 *If X and Y are statistically independent and initial information about X and Y is given as*

$$\underline{p}_i \leq \Pr\{X \leq \alpha_i\} \leq \overline{p}_i, \ \underline{q}_j \leq \Pr\{Y \leq \beta_j\} \leq \overline{q}_j, \ i = 1, ..., n, \ j = 1, ..., m,$$

and $\forall i \leq k$, $\forall j \leq l$ it is valid that $\underline{p}_i \leq \overline{p}_k$ and $\underline{q}_i \leq \overline{q}_l$, then there hold

$$\underline{R} = \sum_{i=1}^{n} (\underline{p}_i - \underline{p}_{i-1})(1 - \overline{q}_{j(i)}), \ j(i) = \min\{j : \alpha_i \leq \beta_j\}, \ \textit{and}$$

$$\overline{R} = 1 - \sum_{k=1}^{m} (\underline{q}_k - \underline{q}_{k-1})(1 - \overline{p}_{l(k)}), \ l(k) = \min\{l : \beta_k \leq \alpha_l\}.$$

Proof. The proof is performed by substituting into (13) and (14) the corresponding boundary values taken from the available set $\underline{p}_i$, $\overline{p}_i$, $\underline{q}_j$ and $\overline{q}_j$, $i = 1, ..., n$, $j = 1, ..., m$, which can be clarified as follows:

$$\underline{R} = \inf_{\underline{p}_i \leq p_i \leq \overline{p}_i, \ \underline{q}_j \leq q_j \leq \overline{q}_j} \underline{R}(p_i, q_j, i \leq n, \ j \leq m) = \sum_{i=1}^{n} (\underline{p}_i - \underline{p}_{i-1})(1 - \overline{q}_{j(i)}),$$

$$\overline{R} = \sup_{\underline{p}_i \leq p_i \leq \overline{p}_i, \ \underline{q}_j \leq q_j \leq \overline{q}_j} \overline{R}(p_i, q_j, i \leq n, \ j \leq m) = 1 - \sum_{k=1}^{m} (\underline{q}_k - \underline{q}_{k-1})(1 - \overline{p}_{l(k)}).$$

■

Corollary 3 *If X and Y are statistically independent and the probability distribution $F_X(x) = \Pr\{X \leq x\}$ of X is known precisely and partial information about Y is given in the form $\underline{q}_j \leq \Pr\{Y \leq \beta_j\} \leq \overline{q}_j$, $j = 1, ..., m$, $\beta_1 \leq ... \leq \beta_m$, then there hold*

$$\underline{R} = \sum_{i=1}^{m} (1 - \overline{q}_i) \left(F_X(\beta_i) - F_X(\beta_{i-1}) \right),$$

$$\overline{R} = 1 - \sum_{k=1}^{m} (\underline{q}_k - \underline{q}_{k-1})(1 - F_X(\beta_k)).$$

Proof. It follows from Theorem 1 and Corollary 1 that

$$\underline{R} = \lim_{n\to\infty} \sum_{i=1}^{n} (p_i - p_{i-1})(1 - \overline{q}_{j(i)}) = \lim_{n\to\infty} \sum_{i=1}^{n} \rho_X(x_i) \triangle x_i (1 - \overline{q}_{j(i)})$$

$$= \sum_{i=1}^{m} \int_{\beta_{i-1}}^{\beta_i} \rho_X(x)(1 - \overline{q}_i)\mathrm{d}x = \sum_{i=1}^{m} (1 - \overline{q}_i)(F_X(\beta_i) - F_X(\beta_{i-1})).$$

The upper reliability can be found similarly. ■

Corollary 4 *If X and Y are statistically independent and the probability distribution $F_Y(y) = \Pr\{Y \leq y\}$ of Y is known precisely and partial information about X is given in the form $\underline{p}_j \leq \Pr\{X \leq \alpha_j\} \leq \overline{p}_j$, $j = 1, ..., n$, $\alpha_1 \leq ... \leq \alpha_n$, then there hold*

$$\underline{R} = \sum_{i=1}^{n} (\underline{p}_i - \underline{p}_{i-1})(1 - F_Y(\alpha_i),$$

$$\overline{R} = 1 - \sum_{i=1}^{n} (1 - \overline{p}_i) \left(F_Y(\alpha_i) - F_Y(\alpha_{i-1}) \right).$$

Proof. Similarly to the proof for Corollary 3. ■

Example 2 *X and Y are independent and their probability distributions are known partially*

$$\Pr\{X \leq \alpha_i\} = p_i,\ \Pr\{Y \leq \beta_j\} = q_j,\ i = 1, 2,\ j = 1, 2,$$

$$\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2.$$

By using Theorem 1, we obtain

$$\underline{R} = p_1(1 - q_1) + (p_2 - p_1)(1 - q_2),$$

$$\overline{R} = 1 - (1 - p_2)q_1.$$

If $p_1 = 0.1$, $p_2 = 0.6$, $q_1 = 0.2$, $q_2 = 0.4$, then $\underline{R} = 0.38$, $\overline{R} = 0.92$.

Now suppose that the random variable Y is governed by the normal distribution with the expectation $a = 10$ and variance $\sigma^2 = 16$. By assuming $\alpha_1 = 6$, $\alpha_2 = 8$, we obtain

$$\underline{R} = p_1(1 - F_Y(\alpha_1)) + (p_2 - p_1)(1 - F_Y(\alpha_2),$$

$$\overline{R} = 1 - (1 - p_1)F_Y(\alpha_1) - (1 - p_2)\left(F_Y(\alpha_2) - F_Y(\alpha_1)\right).$$

Note that $F_Y(\alpha_i) = \Phi\left((\alpha_i - a)/\sigma\right) = \Phi\left((\alpha_i - 10)/4\right)$, $\Phi(x)$ is the standard normal distribution function (Feller 1971). Then

$$\underline{R} = 0.1 \cdot \Phi\left(1\right) + 0.5 \cdot \Phi\left(0.5\right) = 0.1 \cdot 0.84 + 0.5 \cdot 0.69 = 0.429,$$

$$\overline{R} = 1 - 0.9 \cdot 0.16 - 0.4\left(0.84 - 0.69\right) = 0.796.$$

It can be seen from the numerical results that additional information about the distribution function of Y reduces bounds for the stress-strength reliability.

3.2 Lack of knowledge about independence of X and Y

As it was mentioned above, it is not necessary to introduce the judgement of independence between the stress and strength if there is no ground to do so. Reliability can be calculated anyway. Yet, when embarking on not employing this judgement, one should be aware of that the result will be obtained at the cost of larger imprecision.

Assume that initial information about X and Y is the same as that for Corollary 2, i.e.

$$\underline{p}_i \leq \Pr\{X \leq \alpha_i\} \leq \overline{p}_i,\ \underline{q}_j \leq \Pr\{Y \leq \beta_j\} \leq \overline{q}_j,\ i = 1, ..., n,\ j = 1, ..., m,$$

and $\forall i \leq n$, $\forall j \leq m$ and $i \leq j$ it is valid that $\underline{p}_i \leq \underline{p}_j$, $\underline{q}_i \leq \underline{q}_j$, $\overline{p}_i \leq \overline{p}_j$ and $\overline{q}_i \leq \overline{q}_j$. The difference from the corollary is that X and Y are not judged to be independent and the state of knowledge can rather be posed as complete ignorance of whether the random variables are dependent or not (the structural judgement behind this is called logical independence).

In the theorem below the following properties of coherent interval-valued probabilities will be used: $\overline{P}(D) = \min\left\{\overline{P}(D), 1-\underline{P}(D^c)\right\}$ and $\underline{P}(D) = \max\left\{\underline{P}(D), 1-\overline{P}(D^c)\right\}$ (Kuznetsov 1991), $\overline{P}(D) = \inf\limits_{D\subset A}\overline{P}(A)$, $\underline{P}(D) = \sup\limits_{D\supset A}\underline{P}(A)$ and $\underline{P}(A\cap B) = \max\{0, \underline{P}(A)+\underline{P}(B)-1\}$ (Walley 1991), where A, B and D are events.

The asterisk notation in $\underline{R}^*$ and $\overline{R}^*$ will mean that the bounds for the structural reliability are obtained based on ignorance about the dependency of X and Y.

Theorem 2 *If X and Y are not judged to be independent, then the lower and upper structural reliabilities $\underline{R}^*$ and $\overline{R}^*$ are calculated as follows:*

$$\underline{R}^* = \max_{i=1,\dots,n}\max\left(0, \underline{p}_i - \overline{q}_{j(i)}\right),\ \ j(i) = \min\{j : \alpha_i \leq \beta_j\},\ \ \textit{and}$$

$$\overline{R}^* = 1 - \max_{k=1,\dots,m}\max\left(0, \underline{q}_k - \overline{p}_{l(k)}\right),\ \ l(k) = \min\{l : \beta_k \leq \alpha_l\}.$$

Proof. Let us introduce notation: D is the event $\{Y > X\}$, A_i is the event $\{x \in [0, \alpha_i]\}$ and A_i^c is the set complement to A_i, B_i is the event $\{y \in [0, \beta_i]\}$, and A_iB_k is a subset of the universal set $A_{n+1}\times B_{m+1}$. According to the properties cited above, there hold

$$\overline{R}^* = \overline{P}(D) = \min\left\{\inf_{i,k:D\subset A_iB_k}\overline{P}(A_iB_k), 1 - \sup_{i,k:D^c\supset A_iB_k}\underline{P}(A_iB_k)\right\}$$

$$= \min\left\{\min_{k=1,\dots,m}\overline{P}\left((A^c_{l(k)}\cap B_k)^c\right), 1 - \max_{k=1,\dots,m}\underline{P}(A^c_{l(k)}\cap B_k)\right\}$$

$$= \min\left\{1 - \max_{k=1,\dots,m}\underline{P}(A^c_{l(k)}\cap B_k), 1 - \max_{k=1,\dots,m}\underline{P}(A^c_{l(k)}\cap B_k)\right\}$$

$$= 1 - \max_{k=1,\dots,m}\max\left(0, \underline{P}(A^c_{l(k)}) + \underline{P}(B_k) - 1\right)$$

$$= 1 - \max_{k=1,\dots,m}\max\left(0, -\overline{P}(A_{l(k)}) + \underline{P}(B_k)\right) = 1 - \max_{k=1,\dots,m}\max\left(0, \underline{q}_k - \overline{p}_{l(k)}\right).$$

$$\underline{R}^* = \underline{P}(D) = \max\left\{\sup_{i,j:D\supset A_i\cap B_j}\underline{P}(A_iB_j), 1 - \inf_{i,j:D^c\subset A_i\cap B_j}\overline{P}(A_iB_j)\right\}$$

$$= \max\left\{\max_{i=1,\dots,n}\underline{P}(A_i\cap B^c_{j(i)}), 1 - \min_{i=1,\dots,n}\overline{P}\left((A_i\cap B^c_{j(i)})^c\right)\right\}$$

$$= \max\left\{\max_{i=1,\dots,n}\underline{P}(A_i\cap B^c_{j(i)}), \max_{i=1,\dots,n}\underline{P}(A_i\cap B^c_{j(i)})\right\}$$

$$= \max_{i=1,\ldots,n} \max\left(0, \underline{P}(A_i) + \underline{P}(B^c_{j(i)}) - 1\right) = \max_{i=1,\ldots,n} \max\left(0, \underline{p}_i - \overline{q}_{j(i)}\right).$$

■

Corollary 5 *If there is a state of ignorance about the dependency of X and Y and their probability distributions are known precisely, then*

$$\underline{R}^* = \max_{z\geq 0} \max\left\{0, F_X(z) - F_Y(z)\right\},$$

$$\overline{R}^* = 1 - \max_{z\geq 0} \max\left\{0, F_Y(z) - F_X(z)\right\}.$$

Here $F_X(x)$ and $F_Y(x)$ are the cumulative distribution functions of the random variables X and Y, i.e. $F_X(x) = \Pr\{X \leq x\}$ and $F_Y(x) = \Pr\{Y \leq x\}$.

Proof. The formulas follow directly from Theorem 2. ■

It also obviously follows from Theorem 2 that if stress X and strength Y are governed by partially known probability distributions in the form $\Pr\{X \leq \alpha_i\} = p_i,\ \Pr\{Y \leq \beta_j\} = q_j,\ i = 1, ..., n,\ j = 1, ..., m,$ $\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_n$ and $\beta_1 \leq \beta_2 \leq ... \leq \beta_m$, then the lower and upper structural reliabilities are computed as follows:

$$\underline{R}^* = \max_{i=1,\ldots,n} \max\left(0, p_i - q_{j(i)}\right),\ j(i) = \min\{j : \alpha_i \leq \beta_j\},\ \text{and}$$

$$\overline{R}^* = 1 - \max_{k=1,\ldots,m} \max\left(0, q_k - p_{l(k)}\right),\ l(k) = \min\{l : \beta_k \leq \alpha_l\}.$$

This means, even though the probabilities $\Pr\{X \leq \alpha_i\} = p_i$ and $\Pr\{Y \leq \beta_j\} = q_j$ are known precisely and the judgement of independence between stress X and strength Y is not introduced, then only imprecise reliability can be found.

Corollary 6 *If there is a state of ignorance about the dependency of X and Y and the probability distribution $F_X(x) = \Pr\{X \leq x\}$ is known precisely and partial information about Y is given in the form $\underline{q}_j \leq \Pr\{Y \leq \beta_j\} \leq \overline{q}_j$, $j = 1, ..., m$, $\beta_1 \leq ... \leq \beta_m$, then there hold*

$$\underline{R}^* = \max_{i=1,\ldots,m} \max\left\{0, F_X(\beta_i) - \overline{q}_i\right\},$$

$$\overline{R}^{*} = 1 - \max_{i=1,\dots,m} \max\left\{0, \underline{q}_i - F_X(\beta_i)\right\}.$$

Proof. The formulas follow directly from Theorem 2. ■

Corollary 7 *If there is a state of ignorance about the dependency of X and Y and the probability distribution $F_Y(y) = \Pr\{Y \leq y\}$ is known precisely and partial information about X is given in the form $\underline{p}_j \leq \Pr\{X \leq \alpha_j\} \leq \overline{p}_j$, $j = 1, ..., n$, $\alpha_1 \leq ... \leq \alpha_n$, then there hold*

$$\underline{R}^{*} = \max_{i=1,\dots,n} \max\left\{0, \underline{p}_i - F_Y(\alpha_i)\right\},$$

$$\overline{R}^{*} = 1 - \max_{i=1,\dots,n} \max\left\{0, F_Y(\alpha_i) - \overline{p}_i\right\}.$$

Proof. The formulas follow directly from Theorem 2. ■

Example 3 *There is a lack of information of whether X and Y are dependent or not and the source data on the reliability distributions are the same as in Example 2, i.e. $p_1 = \underline{p}_1 = \overline{p}_1 = 0.1$, $p_2 = \underline{p}_2 = \overline{p}_2 = 0.6$, $q_1 = \underline{q}_1 = \overline{q}_1 = 0.2$, $q_2 = \underline{q}_2 = \overline{q}_2 = 0.4$ and $\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2$. Then*

$$\underline{R}^{*} = \max\left(0, \underline{p}_1 - \overline{q}_1, \underline{p}_2 - \overline{q}_2\right) = \max(0, -0.1, 0.2) = 0.2.$$

$$\overline{R}^{*} = 1 - \max\left(0, \underline{q}_1 - \overline{p}_2\right) = 1 - \max(0, -0.4) = 1.$$

As it is seen, the imprecision $\overline{R}^{*} - \underline{R}^{*} = 1 - 0.2 = 0.8$ is larger compared to the case considered in Example 2, which is a consequence of not using the judgement of independence. The same conclusion can be made the case below.

Now suppose that the random variable Y is governed by the normal distribution $N(10, 16)$ (see Example 2). Then

$$\underline{R}^{*} = \max\left\{0,\ \underline{p}_1 - F_Y(\alpha_1),\ \underline{p}_2 - F_Y(\alpha_2)\right\}$$

$$= \max\{0,\ 0.1 - 0.16,\ 0.6 - 0.31\} = 0.29,$$

$$\overline{R}^{*} = 1 - \max\{0,\ F_Y(\alpha_1) - \overline{p}_1,\ F_Y(\alpha_2) - \overline{p}_2\}$$

$$= 1 - \max\{0,\ 0.16 - 0.1,\ 0.31 - 0.6\} = 0.94.$$

4 Known moments of probability distributions

4.1 Independent X and Y

One of the particularities of the current approach is the ability of obtaining the upper and lower bounds of the structural reliability without having to have the probability distributions (precise or partial). Source statistical partial information concerning random variables X and Y may exist in the form of the (precise or imprecise) moments of the probability distributions. Such initial data are less specific ("more" partial) compared to knowing the probability distributions, and one should expect greater imprecision in resultant reliabilities.

Let $\underline{a}$ and $\overline{a}$ are lower and upper bounds of the first moment for the stress and $\underline{b}$ and $\overline{b}$ are the bounds of the first moments for the strength. Here we have to assume that the stress and strength are limited by some finite values T_X and T_Y. Otherwise (see Corollary 8 below), we will arrive at the vacuous reliability $\overline{R} = 1$ and $\underline{R} = 0$.

Suppose first that random variables X and Y are independent. In this case the natural extension for computing the stress-strength reliability is of the form:

$$\underline{R}\left(\overline{R}\right) = \inf_{\mathcal{P}} \left(\sup_{\mathcal{P}}\right) \int_{\mathbf{R}_+} \int_{\mathbf{R}_+} I_{[0,\infty)}(y - x)\rho_X(x)\rho_Y(y)\mathrm{d}x\mathrm{d}y, \tag{15}$$

$$\int_{\mathbf{R}_+} \rho_X(x)\mathrm{d}x = 1,\ \rho_X(x) \geq 0,\ \int_{\mathbf{R}_+} \rho_Y(y)\mathrm{d}y = 1,\ \rho_Y(y) \geq 0,$$

$$\underline{a} \leq \int_{\mathbf{R}_+} x\rho_X(x)\mathrm{d}x \leq \overline{a},\ \underline{b} \leq \int_{\mathbf{R}_+} y\rho_Y(y)\mathrm{d}y \leq \overline{b},\ x \leq T_X,\ y \leq T_Y. \tag{16}$$

This optimisation problem is non-linear and there are no standard procedures to infer explicit analytical solutions. Nevertheless, it is possible to prove the results stated in the following theorem.

Theorem 3 *If stress $X \leq T_X$ and strength $Y \leq T_Y$ are independent and their expectations are interval-valued, that is, $\underline{a} \leq M(X) \leq \overline{a}$ and $\underline{b} \leq M(Y) \leq \overline{b}$, then*

$$\underline{R} = \max\{0, (\underline{b} - \overline{a})/T_Y\}, \tag{17}$$

$$\overline{R} = \min\{1, 1 - (\underline{a} - \overline{b})/T_X\}. \tag{18}$$

Proof. Let us fix some value y_i of the variable y in (15)-(16) and consider the following optimization problems:

$$\underline{R}(y_i)\ \left(\overline{R}(y_i)\right) = \inf_{\rho_X}\left(\sup_{\rho_X}\right)\int_{\mathbf{R}_+} I_{[0,\infty)}(y_i - x)\rho_X(x)\mathrm{d}x,$$

$$\int_{\mathbf{R}_+}\rho_X(x)\mathrm{d}x = 1,\ \rho_X(x) \geq 0,\ \underline{a} \leq \int_{\mathbf{R}_+} x\rho_X(x)\mathrm{d}x \leq \overline{a},\ x \leq T_X.$$

First find $\overline{R}(y_i)$. The corresponding dual problem is of the form:

$$\overline{R}(y_i) = \inf_{c_0,c,d}\left(c_0 + c\overline{a} - d\underline{a}\right),$$

subject to c, $d \in \mathbf{R}^+$, $c_0 \in \mathbf{R}$ and $\forall x \leq T_X$, $c_0 + (c-d)\,x \geq I_{[0,\infty)}(y_i - x)$.

Let us take a few fixed values of x that will help find c, d and c_0 delivering inf to the objective function. By assuming first $T_Y \leq T_X$ and substituting $x = y_i$, $x = T_X$ and $x = 0$ into the above constraint, it can be written

$$c_0 + (c-d)\,y_i \geq 1,\ c_0 + (c-d)\,T_X \geq 0,\ c_0 \geq 1.$$

Since $y_i < T_X$, then the first and second inequality suggest that $c - d \leq 0$, because if $c - d > 0$, then we obtain the contradiction $1 \leq 0$. According to (Gurov & Utkin 1999), in this case $c = 0$ and we have more simple constraints $c_0 - dx \geq I_{[0,\infty)}(y_i - x)$. It can be also concluded that all possible constraints obtained by substituting different values x into the above inequality follow from the three ones:

$$c_0 - dy_i \geq 1,\ c_0 - dT_X \geq 0,\ c_0 \geq 1.$$

Thus, we have the optimization problem with two variables: c_0 and d. By using the graphical method of solving linear optimisation problems we obtain for $y_i \leq \underline{a}$:

$$c_0 = T_X/(T_X - y_i),\ d = 1/(T_X - y_i),$$

and for $y_i > \underline{a}$: $c_0 = 1$ and $d = 0$. As a result,

$$\overline{R}(y_i) = \begin{cases} (T_X - \underline{a})/(T_X - y_i), & y_i \leq \underline{a} \\ 1, & y_i > \underline{a} \end{cases}.$$

In the case $T_Y > T_X$ and $y_i > T_X$, we have $c_0 - dT_X \geq 1$ instead of $c_0 - dT_X \geq 0$. This case gives $\overline{R}(y_i) = 1$.

Similarly, we can obtain the lower bound $\underline{R}(y_i)$. The corresponding dual optimization problem is

$$\underline{R}(y_i) = \sup_{c_0,c,d} (c_0 + c\underline{a} - d\overline{a}),$$

subject to $c,\ d \in \mathbf{R}^+$, $c_0 \in \mathbf{R}$ and $\forall x \leq T_X,\ c_0 + (c-d)\,x \leq I_{[0,\infty)}(y_i - x)$. The above constraints are reduced to the following:

$$c_0 - dy_i \leq 0,\ c_0 - dT_X \leq 0,\ c_0 \leq 1,$$

and

$$\underline{R}(y_i) = \begin{cases} 0, & y_i < \overline{a} \\ 1 - \overline{a}/y_i, & y_i \geq \overline{a} \end{cases}.$$

Now it can be written

$$\underline{R}\left(\overline{R}\right) = \inf_{\rho_Y} \left(\sup_{\rho_Y}\right) \int_{\mathbf{R}_+} \underline{R}(y)\left(\overline{R}(y)\right)\rho_Y(y)\mathrm{d}y,$$

$$\int_{\mathbf{R}_+} \rho_Y(y)\mathrm{d}y = 1,\ \rho_Y(y) \geq 0,\ \underline{b} \leq \int_{\mathbf{R}_+} y\rho_Y(y)\mathrm{d}y \leq \overline{b},\ y \leq T_Y.$$

Let us find $\overline{R}$ first. The corresponding dual problem is of the form:

$$\overline{R} = \inf_{c_0,c,d} \left(c_0 + c\overline{b} - d\underline{b}\right),$$

subject to $c,\ d \in \mathbf{R}^+$, $c_0 \in \mathbf{R}$ and $\forall y \leq T_Y,\ c_0 + (c-d)\,y \geq \overline{R}(y)$. It follows from the constraints

$$c_0 \geq 1 - \underline{a}/T_X,\ c_0 + (c-d)\,\underline{a} \geq 1,\ c_0 + (c-d)\,T_Y \geq 1,$$

that $c - d \geq 0$ and $d = 0$. Since $\underline{a} \leq T_Y$, then the constraint $c_0 + (c-d)\,T_Y \geq 1$ follows from $c_0 + (c-d)\,\underline{a} \geq 1$. Then from all the constraints only two ones

$$c_0 \geq 1 - \underline{a}/T_X,\ c_0 + c\underline{a} \geq 1,$$

form a feasible region together with a vertex of the simplex defined from the equation $1 - \underline{a}/T_X = 1 - c\underline{a}$. Hence one of the solutions is $c = 1/T_X$, $c_0 = 1 - \underline{a}/T_X$, and $\overline{R} = \min\{1, 1 - (\underline{a} - \overline{b})/T_X\}$. Similarly, we can obtain the lower reliability $\underline{R} = \max\{0, (\underline{b} - \overline{a})/T_Y\}$. ■

Corollary 8 *If source information is the expected value of independent stress X and strength Y, and $T_X \to \infty$ and $T_Y \to \infty$, then $\overline{R} = 1$ and $\underline{R} = 0$.*

Proof. The result is obtained by substituting $T_X = \infty$ and $T_Y = \infty$ into (17) and (18). ∎

Corollary 8 implies that the expectations of unbounded stress and strength do not bear any useful information with respect to the probability $P(X \leq Y)$.

Suppose that we know the lower $\underline{a}$ and upper $\overline{a}$ m-th moments of stress and the lower $\underline{b}$ and upper $\overline{b}$ m-th moments of strength, i.e.

$$\underline{a} \leq M(x^m) \leq \overline{a},\ \underline{b} \leq M(y^m) \leq \overline{b}.$$

Theorem 4 *If source information is the interval-valued m-th moment of independent stress X and strength Y , then there hold*

$$\underline{R} = \max\{0, (\underline{b} - \overline{a})/T_Y^m\},$$

$$\overline{R} = \min\{1, 1 - (\underline{a} - \overline{b})/T_X^m\}.$$

Proof. Denote $v = x^m$ and $w = y^m$ and rewrite the constraints $\underline{a} \leq \int_{\mathbf{R}_+} x^m \rho_X(x)dx \leq \overline{a}$ and $x \leq T_X$ as $\underline{a} \leq \int_{\mathbf{R}_+} v\rho_V(v)dv \leq \overline{a}$ and $v \leq T_X^m$, where $\rho_V(v)$ is a new probability density function. The same can be done for the constraints corresponding to the variable Y.

Since there holds $I_{[0,\infty)}(y - x) = I_{[0,\infty)}(w - v)$, then (15)-(16) can be rewritten

$$\underline{R}\,(\overline{R}) = \inf_{\mathcal{P}} \left(\sup_{\mathcal{P}}\right) \int_{\mathbf{R}_+} \int_{\mathbf{R}_+} I_{[0,\infty)}(w - v)\rho_V(v)\rho_W(w)\mathrm{d}v\mathrm{d}w,$$

$$\int_{\mathbf{R}_+} \rho_V(v)\mathrm{d}v = 1,\ \rho_V(v) \geq 0,\ \int_{\mathbf{R}_+} \rho_W(w)\mathrm{d}w = 1,\ \rho_W(w) \geq 0,$$

$$\underline{a} \leq \int_{\mathbf{R}_+} v\rho_V(v)\mathrm{d}v \leq \overline{a},\ \underline{b} \leq \int_{\mathbf{R}_+} w\rho_W(w)\mathrm{d}w \leq \overline{b},\ v \leq T_X^m,\ w \leq T_Y^m.$$

Thus, we have obtained the same problems as for Theorem 3, but T_X and T_Y are replaced by T_X^m and T_Y^m. This implies that the results of Theorem 3 can be extended to the case of the m-th interval-valued moments. ∎

4.2 Lack of knowledge about independence of X and Y

Now consider the case when there is no information about the independence of random variables X and Y. In this case the natural extension of lower and upper first moments of X and Y on the stress-strength can be written as

$$\overline{R} = \inf_{c_0,c_i,d_i} \left(c_0 + c_1\overline{a} - d_1\underline{a} + c_2\overline{b} - d_2\underline{b} \right), \tag{19}$$

subject to c_i, $d_i \in \mathbf{R}^+$, $c_0 \in \mathbf{R}$, $i = 1, 2$, and $\forall (x, y) \in [0, T_X] \times [0, T_Y]$,

$$c_0 + (c_1 - d_1)x + (c_2 - d_2)y \geq I_{[0,\infty)}(y - x), \tag{20}$$

$$\underline{R} = \sup_{c_0,c_i,d_i} \left(c_0 + c_1\underline{a} - d_1\overline{a} + c_2\underline{b} - d_2\overline{b} \right), \tag{21}$$

subject to c_i, $d_i \in \mathbf{R}^+$, $c_0 \in \mathbf{R}$, $i = 1, 2$, and $\forall (x, y) \in [0, T_X] \times [0, T_Y]$,

$$c_0 + (c_1 - d_1)x + (c_2 - d_2)y \leq I_{[0,\infty)}(y - x), \tag{22}$$

Theorem 5 *If there is no information about the independence of stress X and strength Y, then*

$$\underline{R}^* = \max\{0, ((\underline{b} - \overline{a})/T_Y)\},$$

$$\overline{R}^* = \min\{1, 1 - (\underline{a} - \overline{b})/T_X\}.$$

Proof. Since the judgement of independence is additional information, then the interval $[\underline{R}, \overline{R}]$ cannot be wider than the interval $[\underline{R}^*, \overline{R}^*]$ or $\underline{R}^* \leq \underline{R}$ and $\overline{R}^* \geq \overline{R}$. Let us show that $\underline{R}^* = \underline{R}$. It follows from Theorem 3 in this case that

$$c_0 = 0,\ c_1 = 0,\ d_1 = 1/T_Y,\ c_2 = 1/T_Y,\ d_2 = 0.$$

We should prove that this solution belongs to a feasible region for problem (21)-(22). By substituting the solution into (22), we obtain

$$-x/T_Y + y/T_Y \leq I_{[0,\infty)}(y - x).$$

Hence $y - x \leq T_Y I_{[0,\infty)}(y - x)$. If $y \geq x$, then $y - x \leq T_Y$. If $y < x$, then $y - x \leq 0$. This implies that the solution belongs to a feasible region. The upper bound is similarly proved. ■

Similar to Theorem 4 it can proven that if source information is an interval-valued m-th moment of stress X and strength Y , then under the condition of ignorance about the independence of stress X and strength Y, there hold

$$\underline{R} = \max\{0, (\underline{b} - \overline{a})/T_Y^m\},$$

$$\overline{R} = \min\{1, 1 - (\underline{a} - \overline{b})/T_X^m\}.$$

5 Probability distributions on nested intervals

Cases analysed in the current section are worth being broken down, first of all, because they illustrate how some special cases of initial partial data can be bring us to the use of possibility distributions. And they, in fact, show that the use of possibility distributions is rather a very particular case encountered in practice.

5.1 Independent X and Y

Consider a case with the following partial information about the probability distributions:

$$\Pr\{\underline{\alpha}_i \leq X \leq \overline{\alpha}_i\} = p_i,\ \Pr\{\underline{\beta}_j \leq Y \leq \overline{\beta}_j\} = q_j,\ i = 1, ..., n,\ j = 1, ..., m, \tag{23}$$

where

$$[\underline{\alpha}_1, \overline{\alpha}_1] \subset [\underline{\alpha}_2, \overline{\alpha}_2] \subset ... \subset [\underline{\alpha}_n, \overline{\alpha}_n],\ [\underline{\beta}_1, \overline{\beta}_1] \subset [\underline{\beta}_2, \overline{\beta}_2] \subset ... \subset [\underline{\beta}_m, \overline{\beta}_m], \tag{24}$$

$$p_1 \leq p_2 \leq ... \leq p_n,\ q_1 \leq q_2 \leq ... \leq q_m.$$

In other words, there are nested intervals $[\underline{\alpha}_i, \overline{\alpha}_i]$ and $[\underline{\beta}_i, \overline{\beta}_i]$ with known probabilities p_i and q_i of finding X and Y inside these intervals.

Theorem 6 *If source partial information about stress X and strength Y is given in the form (23)-(24),* *then*

$$\underline{R} = \sum_{i=1}^{n}(p_i - p_{i-1})q_{j(i)},\ j(i) = \max\{j : \overline{\alpha}_i \leq \underline{\beta}_j\}\ \textit{and}$$

$$\overline{R} = 1 - \sum_{k=1}^{m}(q_k - q_{k-1})p_{l(k)},\ l(k) = \max\{l : \overline{\beta}_k \leq \underline{\alpha}_l\}.$$

Proof. Let us write down the initial optimisation problem (it is similar to (10)-(12) but with updated constraints)

$$\underline{R}(\overline{R}) = \inf\,(\sup) \sum_{k=1}^{n+1}\sum_{j=1}^{m+1} I_{[0,\infty)}(y_j - x_k)c_k d_j, \tag{25}$$

subject to

$$\sum_{k=1}^{n+1} c_k = 1,\ \sum_{j=1}^{m+1} d_i = 1,$$

$$\sum_{i=1}^{n+1} I_{[\underline{\alpha}_k,\overline{\alpha}_k]}(x_i)c_i = p_k,\ \sum_{i=1}^{m+1} I_{[\underline{\beta}_j,\overline{\beta}_j]}(y_i)d_i = q_j,\ k = 1,...,n,\ j = 1,...,m. \tag{26}$$

In a similar way as it was done in Theorem 1, it can be shown that for all possible k values x_k and y_k, delivering the optima to the objective function (25), meet the following conditions: $x_k \in [\underline{\alpha}_k, \overline{\alpha}_k]\backslash[\underline{\alpha}_{k-1}, \overline{\alpha}_{k-1}]$ and $y_k \in [\underline{\beta}_k, \overline{\beta}_k]\backslash[\underline{\beta}_{k-1}, \overline{\beta}_{k-1}]$. From these conditions and from constraints (26) it can be concluded that

$$c_1 = p_1,\ c_1 + c_2 = p_2, ..., \sum_{i=1}^{n} c_i = p_n,$$

$$d_1 = q_1,\ d_1 + d_2 = q_2, ..., \sum_{i=1}^{m} d_i = q_m.$$

Hence

$$c_k = p_k - p_{k-1},\ d_j = q_j - q_{j-1},\ k = 1,...,n,\ j = 1,...,m.$$

Note that the objective function (25) achieves its minimum if for all $k \leq n+1$ and $j \leq m+1$ there hold $I_{[0,\infty)}(y_j - x_k) = 0$. However, there exist values j and k such that $I_{[0,\infty)}(y_j - x_k) = 1$ for some combinations of y_j and x_k.

Let $j(k)$ be a maximal number j such that there hold $x_k \in [\underline{\alpha}_k, \overline{\alpha}_k]\backslash[\underline{\alpha}_{k-1}, \overline{\alpha}_{k-1}]$, $y_{j(k)} \in [\underline{\beta}_{j(k)}, \overline{\beta}_{j(k)}]\backslash[\underline{\beta}_{j(k)-1}, \overline{\beta}_{j(k)-1}]$ and $\overline{\alpha}_k \leq \underline{\beta}_{j(k)}$. Then $I_{[0,\infty)}(y_j - x_k) = 1$ for all $j \leq j(k) + 1$. Thus, it can be concluded

$$\underline{R} = \sum_{k=1}^{n+1} \sum_{j=1}^{j(k)+1} c_k d_j = \sum_{k=1}^{n+1} \sum_{j=1}^{j(k)+1} (p_k - p_{k-1})(q_j - q_{j-1}).$$

Taking into account that $\sum_{j=1}^{j(k)+1}(q_j - q_{j-1}) = q_{j(k)}$, the last formula is reduced to

$$\underline{R} = \sum_{k=1}^{n} (p_k - p_{k-1}) q_{j(k)}.$$

The upper bound $\overline{R}$ can be computed similarly. ∎

Corollary 9 *If source information about X and Y is given as*

$$\underline{p}_i \leq \Pr\{\underline{\alpha}_i \leq X \leq \overline{\alpha}_i\} \leq \overline{p}_i,\ \underline{q}_j \leq \Pr\{\underline{\beta}_j \leq Y \leq \overline{\beta}_j\} \leq \overline{q}_j,\ i = 1, ..., n,\ j = 1, ..., m,$$

and is consistent in the sense that the larger interval, the higher probability, then there hold

$$\underline{R} = \sum_{i=1}^{n} (\underline{p}_i - \underline{p}_{i-1}) \underline{q}_{j(i)},\ j(i) = \max\{j : \overline{\alpha}_i \leq \underline{\beta}_j\} \text{ and} \tag{27}$$

$$\overline{R} = 1 - \sum_{k=1}^{m} (\underline{q}_k - \underline{q}_{k-1}) \underline{p}_{l(k)},\ l(k) = \max\{l : \overline{\beta}_k \leq \underline{\alpha}_l\}. \tag{28}$$

Proof. Obviously from Theorem 6. ∎

5.2 Lack of knowledge about independence of X and Y

Theorem 7 *If source information about X and Y is given as*

$$\underline{p}_i \leq \Pr\{\underline{\alpha}_i \leq X \leq \overline{\alpha}_i\} \leq \overline{p}_i,\ \underline{q}_j \leq \Pr\{\underline{\beta}_j \leq Y \leq \overline{\beta}_j\} \leq \overline{q}_j,\ i = 1, ..., n,\ j = 1, ..., m,$$

and is consistent in the sense that the larger interval, the higher probability, then there hold

$$\underline{R}^* = \max_{i=1,\dots,n} \max\left(0, \underline{p}_i + \underline{q}_{j(i)} - 1\right),\ j(i) = \max\{j : \overline{\alpha}_i \leq \underline{\beta}_j\} \text{ and} \tag{29}$$

$$\overline{R}^* = 1 - \max_{k=1,\dots,m} \max\left(0, \underline{q}_k + \underline{p}_{l(k)} - 1\right),\ l(k) = \max\{l : \overline{\beta}_k \leq \underline{\alpha}_l\}. \tag{30}$$

Proof. Similarly to the proof of Theorem 2. ∎

¿From this theorem it obviously follows that if $\Pr\{\underline{\alpha}_i \leq X \leq \overline{\alpha}_i\} = p_i,\ \Pr\{\underline{\beta}_j \leq Y \leq \overline{\beta}_j\} = q_j,\ i = 1, ..., n,\ j = 1, ..., m$, then

$$\underline{R}^{*}=\max_{i=1,\ldots,n}\max\left(0,p_{i}+q_{j(i)}-1\right),\ j(i)=\max\{j:\overline{\alpha}_{i}\leq\underline{\beta}_{j}\}\text{ and}$$

$$\overline{R}^{*}=1-\max_{k=1,\ldots,m}\max\left(0,q_{k}+p_{l(k)}-1\right),\ l(k)=\max\{l:\overline{\beta}_{k}\leq\underline{\alpha}_{l}\}.$$

It is seen from Corollary 9 and Theorem 7 that the bounds for stress-strength reliability depend only on the lower bounds for the probabilities of $\Pr\{\underline{\alpha}_i \leq X \leq \overline{\alpha}_i\}$ and $\Pr\{\underline{\beta}_j \leq Y \leq \overline{\beta}_j\}$. According to (Dubois & Prade 1992), upper probabilities induced by a set of lower bounds $\{P(A_i) \geq \gamma_i, i = 1, ..., n\}$ is a possibility measure if the set $\{A_1, ..., A_n\}$ is nested, that is, $A_1 \subseteq A_2 \subseteq ... \subseteq A_n$. The upper probability in this case coincides with the necessity of an event A_k, i.e. $\pi_k = 1 - \gamma_k$. Denote

$$\pi_X(\underline{\alpha}_i) = \pi_X(\overline{\alpha}_i) = 1 - \underline{p}_i,\ i = 1, ..., n,$$

$$\pi_Y(\underline{\beta}_j) = \pi_Y(\overline{\beta}_j) = 1 - \underline{q}_j,\ j = 1, ..., m.$$

Then X and Y can be regarded as fuzzy variables with the membership functions $\pi_X(\underline{\alpha}_i) = \pi_X(\overline{\alpha}_i)$ and $\pi_Y(\underline{\beta}_j) = \pi_Y(\overline{\beta}_j)$, $i = 1, ..., n$, $j = 1, ..., m$. Let us prove that the bounds for stress-strength reliability can be regarded as the possibility and necessity of the event that the fuzzy variable Y is greater than the fuzzy variable X, that is, $\overline{R} = \Pi\{Y > X\}$ and $\underline{R} = 1 - \Pi\{Y \leq X\} = N\{Y > X\}$. Here Π is possibility and N is necessity.

Theorem 8 *The upper stress-strength reliability $\overline{R}$ defined from (28) or from (30) is the possibility measure $\Pi\{Y > X\}$. The lower stress-strength reliability $\underline{R}$ defined from (27) or from (29) is the necessity measure $N\{Y > X\}$.*

Proof. Let us prove that either $\underline{R} = 0$ or $\overline{R} = 1$. Let K and L be sets of indices such that if $k \in K$, then there holds $\overline{\alpha}_k \leq \underline{\beta}_{j(k)}$, and if $k \in L$, then there holds $\overline{\beta}_k \leq \underline{\alpha}_{l(k)}$. If $K = \emptyset$, then it follows from (27) and (29) that $\underline{R} = 0$. If $L = \emptyset$, then it follows from (28) and (30) that $\overline{R} = 1$. It is necessary to prove that if $K \neq \emptyset$, then $L = \emptyset$.

Let $K \neq \emptyset$, then $k = 1 \in K$. Indeed, $\overline{\alpha}_1 \leq \overline{\alpha}_i$ for $i = 2, ..., n$. Therefore, if there exists $\overline{\alpha}_k \leq \underline{\beta}_{j(k)}$, then there holds $\overline{\alpha}_1 \leq \underline{\beta}_{j(k)}$. Moreover, there holds $\overline{\alpha}_1 \leq \underline{\beta}_1$ because $\underline{\beta}_1 \geq \underline{\beta}_j$ for $j = 2, ..., m$. We can similarly prove that if $L \neq \emptyset$, then $\overline{\beta}_1 \leq \underline{\alpha}_1$. So, we have a contradiction. Consequently, either $\underline{R} = 0$ or $\overline{R} = 1$. Let A be the event $Y > X$. It is obvious that

$$\Pi\left\{A \cup A^c\right\} = 1 = \max(\overline{R}, 1 - \underline{R}) = 1 = \max\left(\Pi\left\{A\right\}, \Pi\left\{A^c\right\}\right),$$

$$N\left\{A \cap A^c\right\} = 0 = \min(\underline{R}, 1 - \overline{R}) = 0 = \min\left(N\left\{A\right\}, N\left\{A^c\right\}\right).$$

Hence, according to (Walley 1996), $\overline{R}$ and $\underline{R}$ are possibility and necessity measures. ■

Thus, if variables X and Y are independent, then

$$\underline{R} = \sum_{i=1}^{n}(\pi_X(\underline{\alpha}_{i-1}) - \pi_X(\underline{\alpha}_i))(1 - \pi_Y(\underline{\beta}_{j(i)})),\ j(i) = \max\{j : \overline{\alpha}_i \leq \underline{\beta}_j\} \text{ and}$$

$$\overline{R} = 1 - \sum_{k=1}^{m}(\pi_Y(\underline{\beta}_{k-1}) - \pi_Y(\underline{\beta}_k))(1 - \pi_X(\underline{\alpha}_{l(k)})),\ l(k) = \max\{l : \overline{\beta}_k \leq \underline{\alpha}_l\}.$$

If there is no information about independence of X and Y, then

$$\underline{R}^* = \max_{i=1,\dots,n} \max\left(0, 1 - \pi_X(\underline{\alpha}_{i-1}) - \pi_Y(\underline{\beta}_{j(i)})\right),$$

$$\overline{R}^* = 1 - \max_{k=1,\dots,m} \max\left(0, 1 - \pi_Y(\underline{\beta}_k) - \pi_X(\underline{\alpha}_{l(k)})\right).$$

6 Summary

The current section gives an overview of all the formulas obtained for the calculation of structural reliability given partial source information. The results are grouped into two tables: (1) stress X and strength Y are judged statistically independent, and (2) there is a lack of knowledge of their dependency. There exist some additional conditions for applying the formulas and which are not shown in the tables. For complete references the reader should see the text of paper.

TABLE I Lower and upper bounds of structural reliability given stress X and strength Y are statistically independent

Source information	**Structural reliability** (Lower and upper bounds)	**Notation and** **additional conditions**
$\Pr\{X \le \alpha_i\} = p_i$ $\Pr\{Y \le \beta_j\} = q_j$	$\underline{R} = \sum_{i=1}^{n} (p_i - p_{i-1})(1 - q_{j(i)})$ $\overline{R} = 1 - \sum_{k=1}^{m} (q_k - q_{k-1})(1 - p_{l(k)})$	$j(i) = \min\{j : \alpha_i \le \beta_j\}$
$\underline{p}_i \le \Pr\{X \le \alpha_i\} \le \overline{p}_i$ $\underline{q}_j \le \Pr\{Y \le \beta_j\} \le \overline{q}_j$	$\underline{R} = \sum_{i=1}^{n} (\underline{p}_i - \underline{p}_{i-1})(1 - \overline{q}_{j(i)})$ $\overline{R} = 1 - \sum_{k=1}^{m} (\underline{q}_k - \underline{q}_{k-1})(1 - \overline{p}_{l(k)})$	$l(k) = \min\{l : \beta_k \le \alpha_l\}$
$F_X(x) = \Pr\{X \le x\}$ is precise $\underline{q}_j \le \Pr\{Y \le \beta_j\} \le \overline{q}_j$	$\underline{R} = \sum_{i=1}^{m} (1 - \overline{q}_i)(F_X(\beta_i) - F_X(\beta_{i-1}))$ $\overline{R} = 1 - \sum_{k=1}^{m} (\underline{q}_k - \underline{q}_{k-1})(1 - F_X(\beta_k))$	
$F_Y(y) = \Pr\{Y \le y\}$ is precise $\underline{p}_j \le \Pr\{X \le \alpha_j\} \le \overline{p}_j$	$\underline{R} = \sum_{i=1}^{n} (\underline{p}_i - \underline{p}_{i-1})(1 - F_Y(\alpha_i))$ $\overline{R} = 1 - \sum_{i=1}^{n} (1 - \overline{p}_i)(F_Y(\alpha_i) - F_Y(\alpha_{i-1}))$	
$\underline{a} \le M(X) \le \overline{a}$ $\underline{b} \le M(Y) \le \overline{b}$	$\underline{R} = \max\{0, (\underline{b} - \overline{a})/T_Y\}$ $\overline{R} = \min\{1, 1 - (\underline{a} - \overline{b})/T_X\}$	$Y \le T_Y$ $X \le T_X$
$M(X)$ & $M(Y)$ precise/imprecise	$\underline{R} = 0$ and $\overline{R} = 1$	$X(Y) \le T_X(T_Y) \to \infty$
$\underline{a} \le M(X^m) \le \overline{a}$ $\underline{b} \le M(Y^m) \le \overline{b}$	$\underline{R} = \max\{0, (\underline{b} - \overline{a})/T_Y^m\}$ $\overline{R} = \min\{1, 1 - (\underline{a} - \overline{b})/T_X^m\}$	$Y \le T_Y$ $X \le T_X$
$\Pr\{\underline{\alpha}_i \le X \le \overline{\alpha}_i\} = p_i$, $\Pr\{\underline{\beta}_j \le Y \le \overline{\beta}_j\} = q_j$	$\underline{R} = \sum_{i=1}^{n} (p_i - p_{i-1}) q_{j(i)}$ $\overline{R} = 1 - \sum_{k=1}^{m} (q_k - q_{k-1}) p_{l(k)}$	$[\underline{\alpha}_1, \overline{\alpha}_1] \subseteq ... \subseteq [\underline{\alpha}_n, \overline{\alpha}_n]$ $[\underline{\beta}_1, \overline{\beta}_1] \subseteq ... \subseteq [\underline{\beta}_n, \overline{\beta}_n]$
$\underline{p}_i \le \Pr\{\underline{\alpha}_i \le X \le \overline{\alpha}_i\} \le \overline{p}_i$ $\underline{q}_j \le \Pr\{\underline{\beta}_j \le Y \le \overline{\beta}_j\} \le \overline{q}_j$	$\underline{R} = \sum_{i=1}^{n} (\underline{p}_i - \underline{p}_{i-1}) \underline{q}_{j(i)}$ $\overline{R} = 1 - \sum_{k=1}^{m} (\underline{q}_k - \underline{q}_{k-1}) \underline{p}_{l(k)}$	$j(i) = \max\{j : \overline{\alpha}_i \le \underline{\beta}_j\}$ $l(k) = \max\{l : \overline{\beta}_k \le \underline{\alpha}_l\}$

TABLE II Lower and upper bounds of structural reliability given a lack of knowledge if stress X and strength Y are statistically independent

Source information	**Structural reliability** (Lower and upper bounds)	**Notation and** **additional conditions**
$\Pr\{X \le \alpha_i\} = p_i$ $\Pr\{Y \le \beta_j\} = q_j$	$\underline{R}^* = \max_{i=1,\dots,n} \max\left(0, p_i - q_{j(i)}\right)$ $\overline{R}^* = 1 - \max_{k=1,\dots,m} \max\left(0, q_k - p_{l(k)}\right)$	$j(i) = \min\{j : \alpha_i \le \beta_j\}$
$\underline{p}_i \le \Pr\{X \le \alpha_i\} \le \overline{p}_i$ $\underline{q}_j \le \Pr\{Y \le \beta_j\} \le \overline{q}_j$	$\underline{R}^* = \max_{i=1,\dots,n} \max\left(0, \underline{p}_i - \overline{q}_{j(i)}\right)$ $\overline{R}^* = 1 - \max_{k=1,\dots,m} \max\left(0, \underline{q}_k - \overline{p}_{l(k)}\right)$	$l(k) = \min\{l : \beta_k \le \alpha_l\}$
$F_X(x) = \Pr\{X \le x\}$ is precise $\underline{q}_j \le \Pr\{Y \le \beta_j\} \le \overline{q}_j$	$\underline{R}^* = \max_{i=1,\dots,m} \max\{0, F_X(\beta_i) - \overline{q}_i\}$ $\overline{R}^* = 1 - \max_{i=1,\dots,m} \max\left\{0, \underline{q}_i - F_X(\beta_i)\right\}$	
$F_Y(y) = \Pr\{Y \le y\}$ is precise $\underline{p}_j \le \Pr\{X \le \alpha_j\} \le \overline{p}_j$	$\underline{R}^* = \max_{i=1,\dots,n} \max\left\{0, \underline{p}_i - F_Y(\alpha_i)\right\}$ $\overline{R}^* = 1 - \max_{i=1,\dots,n} \max\{0, F_Y(\alpha_i) - \overline{p}_i\}$	
$F_X(x) = \Pr\{X \le x\}$ is precise $F_Y(y) = \Pr\{Y \le y\}$ is precise	$\underline{R}^* = \max_{z \ge 0} \max\{0, F_X(z) - F_Y(z)\}$ $\overline{R}^* = 1 - \max_{z \ge 0} \max\{0, F_Y(z) - F_X(z)\}$	
$\underline{a} \le M(X) \le \overline{a}$ $\underline{b} \le M(Y) \le \overline{b}$	$\underline{R}^* = \max\{0, (\underline{b} - \overline{a})/T_Y\}$ $\overline{R}^* = \min\{1, 1 - (\underline{a} - \overline{b})/T_X\}$	$Y \le T_Y$ $X \le T_X$
$\underline{a} \le M(X^m) \le \overline{a}$ $\underline{b} \le M(Y^m) \le \overline{b}$	$\underline{R} = \max\{0, (\underline{b} - \overline{a})/T_Y^m\}$ $\overline{R} = \min\{1, 1 - (\underline{a} - \overline{b})/T_X^m\}$	$Y \le T_Y$ $X \le T_X$
$\Pr\{\underline{\alpha}_i \le X \le \overline{\alpha}_i\} = p_i$, $\Pr\{\underline{\beta}_j \le Y \le \overline{\beta}_j\} = q_j$	$\underline{R}^* = \max_{i=1,\dots,n} \max\left(0, p_i + q_{j(i)} - 1\right)$ $\overline{R}^* = 1 - \max_{k=1,\dots,m} \max\left(0, q_k + p_{l(k)} - 1\right)$	$[\underline{\alpha}_1, \overline{\alpha}_1] \subseteq \dots \subseteq [\underline{\alpha}_n, \overline{\alpha}_n]$ $[\underline{\beta}_1, \overline{\beta}_1] \subseteq \dots \subseteq [\underline{\beta}_n, \overline{\beta}_n]$
$\underline{p}_i \le \Pr\{\underline{\alpha}_i \le X \le \overline{\alpha}_i\} \le \overline{p}_i$ $\underline{q}_j \le \Pr\{\underline{\beta}_j \le Y \le \overline{\beta}_j\} \le \overline{q}_j$	$\underline{R}^* = \max_{i=1,\dots,n} \max\left(0, \underline{p}_i + \underline{q}_{j(i)} - 1\right)$ $\overline{R}^* = 1 - \max_{k=1,\dots,m} \max\left(0, \underline{q}_k + \underline{p}_{l(k)} - 1\right)$	$j(i) = \max\{j : \overline{\alpha}_i \le \underline{\beta}_j\}$ $l(k) = \max\{l : \overline{\beta}_k \le \underline{\alpha}_l\}$

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