Reducing the Pareto optimal set in MCDM using imprecise probabilities

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Abstract: An approach for reducing a set of Pareto optimal solutions on the basis of specific information about importance of criteria is proposed in the paper. The DM's judgments about criteria have a clear behavior interpretation and can be used in various decision problems. It is shown that the imprecise probability theory can be successfully applied for formalizing the available information which is represented by means of a set of probability measures. Simple explicit expressions instead of linear programming problems are derived for dealing with three decision rules: maximality, interval dominance and interval bound dominance rules. Numerical examples illustrate the proposed approach.

Keywords: multi-criteria decision making; imprecise probabilities; desirable gambles; sets of probability measures; judgments; preferences; Pareto set.

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1 Introduction

A lot of methods for solving *multi-criteria decision making* (MCDM) problems are based on combining or aggregating of *decision criteria*. According to these methods, *decision alternatives* (DA's) are compared by using an aggregated criterion. There are different ways for criteria combining. Widely-spread ways are linear, multiplicative and maximin combinations (Keeney and Raiffa (1976); Saaty (1980)). For instance, the well-known analytic hierarchy process method proposed by Saaty (1980) is based on the linear combination of criteria. However, in spite of the popularity of the aggregation methods for solving MCDM problems, they do not have a strong justification at times. This difficulty takes place very often when we have only partial information about weights of criteria.

Another part of methods is not directly based on the combining of criteria. The corresponding methods are based on reducing the so-called Pareto set of non-dominated solutions by exploiting some additional information about importance of criteria provided by experts, decision makers (DM's), etc. The amount of the additional information and its consistency determine the number of DA's in a reduced Pareto set. Ideally, the reduced Pareto set should consist of a single DA.

Procedures for processing the additional information and for reducing the Pareto set totally depend on the type of available data or judgments. Many authors use the "weights" of criteria $\mathbf{v} = (v_1, ..., v_r)$ and different kinds of their ranking. For instance, Park and Kim (1997), Kim and Ahn (1999) distinguish between the following approaches to the elicitation of attribute weights: weak ranking ($v_i \geq v_j$); strict ranking ($v_i - v_j \geq \lambda_i$); ranking with multiples ($v_i \geq \lambda_i v_j$); interval form: ($\lambda_i \leq v_i \leq \lambda_i + \epsilon_i$); ranking of differences ($v_i - v_j \geq v_k - v_l$). Here $\lambda_i \geq 0$, $\epsilon_i \geq 0$. In fact, the above information about the weights of criteria can be regarded as imprecise or incomplete. It should be noted that a lot of methods and approaches have been developed and proposed to solve multi-criteria decision making problems under imprecise and incomplete information about criteria and (or) decision alternatives and to model the preference information (Arora and Arora (2010); Das et al. (2012); Dellnitz and Witting (2009); Frikha et al. (2010); Raut et al. (2012)). One of the pioneering works (Weber (1987)) provides a general framework for decision making with incomplete information, where the incomplete information about states of nature and utilities is formalized by means of probability intervals and linear inequalities, respectively. The proposed framework leads to solving the linear programming problems. Various extensions of the framework taking into account some peculiarities of eliciting the decision information have been provided by many authors. Danielson et al. (2007); Ekenberg and Thorbioernson (2001) proposed a class of second-order uncertainty models applied to decision making under incomplete information. Approaches for constructing sets of weights of criteria can be found in works (Mustajoki et al. (2006, 2005); Tervonen et al. (2004)). In particular, the expert opinions in the form of the preference ratios have been studied by Salo and Hamalainen (2001). An interesting method for the analysis of incomplete preference information in hierarchical weighting models of the multi-criteria decision making leading to possibly non-convex sets of feasible attribute weights has been proposed by Salo and Punkka (2005). A novel belief function reasoning approach to the MCDM problem under uncertainty has been proposed by Fan and Nguyen (2011). The decision making problems with multiple decision makers have been studied by Velazquez et al. (2010). Methods for solving MCDM problems with the fuzzy initial information have been considered by Mahata and Goswami (2009); Sakawa and Nishizak (2012); Thipparat et al. (2009)

Another very interesting type of judgments elicited from DM's or experts for reducing the Pareto optimal set has been proposed by Noghin (1997, 2002) as the theory of relative importance of criteria. Some details of the theory will be considered below. This type of judgments does not require to have identical numerical scales for criteria. It has a simple and clear behavior interpretation. Moreover, it is very simple from the computation point of view.

It turns out that the Noghin's theory can be considered in the framework of imprecise probability theory (Walley (1991)) by applying the so-called desirable gambles (Walley (1991, 2000)). In particular, Noghin's decision rule totally coincides with Walley's maximality rule Walley (1991). This peculiarity has been indicated by Utkin (2009) where one decision rule (Walley's maximality rule) was exploited for decision making. However, the approach can be extended on several special decision rules which are used in imprecise probability theory. The main idea for the extension is to construct a set of probability measures produced by the judgments about relative importance of criteria and to make decisions in accordance with the set. Therefore, new extensions of Noghin's theory are studied in the paper, including interval dominance rule (Zaffalon et al. (2003)) and interval bound dominance rule which was proposed by Destercke (2010). It is important to point out here that sets of probability measures are not sets of weights which are used in many aforementioned works (see, for example, (Kim and Ahn (1999); Park and Kim (1997); Salo and Punkka (2005))). They have a quite different meaning.

The paper is organized as follows. The main definitions of MCDM and elements of Noghin's theory of relative importance of criteria are provided in Section 2. Noghin's theory is formulated in the framework of desirable gambles in Section 3. Moreover, three decision rules are studied in this section based on simple comparative judgments. Variants of sets of relative importance judgments applied to three decision rules are investigated in Section 4. Numerical examples illustrate the proposed methods.

2 The MCDM problem statement and Noghin's relative importance of criteria

A general MCDM problem can be formulated in the following way. Suppose that there is a set of DA's $\mathbb{X} = \{X_1, ..., X_n\}$ consisting of n elements. Moreover, there is a set of criteria $\mathbb{C} = \{C_1, ..., C_r\}$ consisting of r elements, $r \geq 2$. For every DA, say the k-th DA, we can write the value of the i-th criterion $C_i(X_k)$ briefly denoted x_{ki}, $k = 1, ..., n$, $i = 1, ..., r$. We will say below that the k-th DA is characterized by the vector $X_k = (x_{k1}, ..., x_{kr})$. We assume that the number of criteria and the number of DA's are finite.

To solve a MCDM problem is to find a set of all optimal solutions denoted by $\text{Opt}\mathbb{X} \subseteq \mathbb{X}$, which can be regarded as the best solutions under certain conditions.

By making decisions, we usually have to take many objectives or criteria into account. The main feature here is that the different objectives are most likely conflicting and the final decision is commonly called a trade-off. When dealing with multiple objectives, solutions can be incomparable since they can dominate each other in different objectives. This lead to the notion of *Pareto optimality*, which is based on a partial order among the solutions. A solution is called Pareto optimal, if it is not dominated by any other solution, that is, if there is no other solution that is better in at least one objective and not worse in any of the other objectives. Naturally, Pareto optimal solutions are the candidates for a trade-off.

Let us give some standard definitions related to Pareto optimal solutions under assumption that there is no information about importance of criteria.

Definition 2.1: *$X \in \mathbb{X}$ dominates $Y \in \mathbb{X}$, denoted $X \succ Y$ iff $\forall i = 1, ..., r$, $x_i \geq y_i$ with at least one strict inequality.*

Definition 2.2: *$Y \in \mathbb{X}$ is a Pareto optimal alternative, also called an efficient alternative, iff $\nexists X \in \mathbb{X}$ such that $X \succ Y$. The set of all Pareto optimal alternatives in $\mathbb{X}$ or Pareto set is denoted $\mathcal{P}(\mathbb{X})$.*

It follows from the above definitions that the following inclusions are valid $\mathrm{Opt}\mathbb{X} \subseteq \mathcal{P}(\mathbb{X}) \subseteq \mathbb{X}$.

For many optimization problems, the number of Pareto optimal solutions can be rather large. Therefore, the problem of reducing Pareto optimal sets by obtaining the additional information is very important.

For reducing the Pareto optimal set, Noghin (1997) proposed the so-called theory of relative importance of criteria. This theory is based on the standard axioms and definitions of Pareto optimal solutions and the property of preference relations. A binary relation $\mathcal{R}$ defined on $\mathbb{R}^r$ is said to be invariant with respect to positive linear transformation if for any vectors $X, Y, c \in \mathbb{R}^r$ and each positive number α the relationship $X\mathcal{R}Y$ implies $(\alpha X + c)\,\mathcal{R}\,(\alpha Y + c)$. It is assumed below that the preference relation $\succ$ is invariant with respect to positive linear transformation.

The main idea of Noghin's theory is to compare criteria by means of parameters.

Definition 2.3: *Let $i, j \in N = \{1, 2, ..., r\}$, $i \neq j$. We say that the i-th criterion is more important than the j-th criterion with two positive parameters w_i and w_j if for any two vectors $X, Y \in \mathbb{X}$ such that*

$$x_i > y_i,\ x_j < y_j,\ x_k = y_k,\ \forall k \in N \backslash \{i, j\},$$
$$x_i - y_i = w_i,\ x_j - y_j = -w_j,$$

the relationship $X \succ Y$ is valid.

A behavior interpretation of the parameters w_i and w_j is the following. The DM is willing to pay w_j units for the j-th criterion in order to get w_i units for the i-th criterion. The relative importance coefficient is defined as

$$\theta_{ij} = \frac{w_j}{w_i + w_j}.$$

It can be seen that $0 < \theta_{ij} < 1$. At that, θ_{ij} is close to 1 if $w_j \gg w_i$. Moreover, θ_{ij} is close to 0 if $w_j \ll w_i$.

Introduce the following vector

$$W_{ij} = (0, ..., 0, w_i, 0, ..., -w_j, 0, ..., 0),$$

whose $r-2$ elements are zero, the i-th element is w_i, the j-th element is $-w_j$. If the relation $X \succ Y$ is valid with the given parameters w_i and w_j, then we can write that the relation $W_{ij} \succ 0_r$ is valid. Here 0_r is the vector of r zero elements. The relation $W_{ij} \succ 0_r$ is equivalent to the relation $\Theta_{ij} \succ 0_r$, where

$$\Theta_{ij} = (0, ..., 0, 1-\theta_{ij}, 0, ..., -\theta_{ij}, 0, ..., 0),$$

or

$$\Theta_{ij} = (0, ..., 0, \theta_{ji}, 0, ..., -\left(1-\theta_{ji}\right), 0, ..., 0),$$

One of the main results of Noghin's theory of the relative importance of criteria is the following his theorem (Noghin (1997)).

Theorem 2.4: *Let the i-th criterion be more important than the j-th criterion with the pair of positive parameters w_i and w_j. Then for any nonempty set of optimal vectors* $\mathrm{Opt}\mathbb{X}$*, it follows that*

$$\mathrm{Opt}\mathbb{X} \subseteq \mathcal{P}^*(\mathbb{X}) \subseteq \mathcal{P}(\mathbb{X}),$$

where $\mathcal{P}(\mathbb{X})$ is a set of Pareto-optimal vectors with respect to criteria $\mathbb{C} = \{C_1, ..., C_r\}$; $\mathcal{P}^*(\mathbb{X})$ is a set of Pareto-optimal vectors with respect to criteria $\mathbb{C}^* = \{C_1^*, ..., C_r^*\}$ such that

$$C_j^* = w_j C_i + w_i C_j,\ C_k^* = C_k,\ k \neq j.$$

In other words, Theorem 2.4 provides a simple computation way for reducing the Pareto optimal set $\mathcal{P}(\mathbb{X})$. Its proof is based on properties of convex cones (Noghin (2002)) produced by preferences of the form $W_{ij} \succ 0_r$. Theorem 2.4 is very important because it is a tool for dealing with the information about the relative importance of criteria. It can be easy rewritten in terms of the relative importance coefficients θ_{ij}.

At the same time, the same results can be obtained in the framework of desirable gambles (Walley (1991, 2000)) by accepting the fact that the judgments about the importance of criteria in the form of vectors W_{ij} or Θ_{ij} produce some set of probability measures called also a credal set (Giron and Rios (1980)) which can be studied by exploiting the imprecise probability theory.

3 Decision rules by simple comparative judgments

There are a number of rules or *global criteria* for making decision in the framework of imprecise probabilities. However, we consider only the rules inducing a partial order, i.e., reducing the Pareto optimal set. These are the maximality rule (Walley (1991)), interval dominance (Zaffalon et al. (2003)) and interval bound dominance proposed by Destercke (Destercke (2010)).

3.1 Desirable gambles

A goal of this section is to consider Noghin's theory of the relative importance of criteria in the framework of desirable gambles (Walley (1991, 2000)) and to show that its results and statements can be rather simply obtained on the basis of the framework. Preliminaries of the framework of desirable gambles given below can be found in (Walley (2000)).

Let Ω denote the set of possible outcomes under consideration. A bounded mapping from Ω to $\mathbb{R}$ (the real numbers) is called a *gamble*. Let $\mathcal{L}$ be a nonempty set of gambles. A mapping $\underline{P} : \mathcal{L} \to \mathbb{R}$ is called a lower prevision or lower expectation. The lower prevision of a gamble X is interpreted as a supremum buying price for X, meaning that it is acceptable to pay any price smaller than $\underline{P}(X)$ for the uncertain reward X. A lower prevision is said to be coherent when it is the lower envelope of some set of linear expectations, i.e., when there is a nonempty set of probability measures, $\mathcal{M}$, such that $\underline{P}(X) = \inf \{E_P(X) : P \in \mathcal{M}\}$ for all $X \in \mathcal{L}$, where $E_P(X)$ denotes the expectation of X with respect to P. The conjugate upper prevision is determined by $\overline{P}(X) = -\underline{P}(-X)$. It is interpreted as an infimum selling price for X.

For $X, Y \in \mathcal{L}$, write $X \geq Y$ to mean that $X(\omega) \geq Y(\omega)$ for all $\omega \in \Omega$, and write $X > Y$ to mean $X \geq Y$ and $X(\omega) > Y(\omega)$ for some $\omega \in \Omega$. According to Walley (1991), a gamble X is *inadmissible* in $\mathcal{L}$ when there is $Y \in \mathcal{L}$ such that $Y \geq X$ and $Y \neq X$. Otherwise X is *admissible* in $\mathcal{L}$. The subset $\mathcal{P}$ of admissible gambles in $\mathcal{L}$ is an analogue of the Pareto set in MCDM. A set of desirable gambles, denoted by $\mathcal{D}$, is a subset of $\mathcal{L}$. A set of desirable gambles is said to be *coherent* when it satisfies the four axioms:

D1. $0 \notin \mathcal{D}$.

D2. if $X \in \mathcal{L}$ and $X > 0$, then $X \in \mathcal{D}$.

D3. if $X \in \mathcal{D}$ and $c \in \mathbb{R}_+$, then $cX \in \mathcal{D}$.

D4. if $X \in \mathcal{D}$ and $Y \in \mathcal{D}$, then $X + Y \in \mathcal{D}$.

Thus a coherent set of desirable gambles is a convex cone of gambles that contains all positive gambles ($X > 0$) but not the zero gamble. Consequence of the axioms: If $X \in \mathcal{D}$ and $Y \geq X$, then $Y \in \mathcal{D}$.

It can be seen from the axioms of coherence that D3 and D4 coincide with the assumed property of preference relations used by Noghin in his theory. Moreover, it can be seen from Definition 2.3 that assessments of the parameters w_i and w_j can be regarded as some extension of the probability ratios studied by Walley (1991). The probability ratios generalize the comparative probability judgments and have the form "A is at least l times as probable as B", where l is a positive number. The gamble $A - lB$ is almost desirable. This implies that $A \succ lB$.

Walley states that there is a one-to-one correspondence between coherent sets of desirable gambles and coherent partial preference orderings, defined by $X \succ Y$ if and only if $X - Y \in \mathcal{D}$. This is very important statement which allows to find the same correspondence between the framework of desirable gambles and Noghin's theory.

If a closed convex set of probability measures $\mathcal{M}$ is given, then we can define a set of desirable gambles as follows:

$$\mathcal{D} = \{X \in \mathcal{L} : X > 0 \text{ or } \mathbb{E}_P(X) > 0, \forall P \in \mathcal{M}\}. \tag{1}$$

Then $\mathcal{D}$ is coherent and $\mathcal{M}$ can be recovered from it by

$$\mathcal{M} = \{P : \mathbb{E}_P(X) \geq 0,\ \forall X \in \mathcal{D}\}. \tag{2}$$

Note that (1) can be rewritten as

$$\mathcal{D} = \{X \in \mathcal{L} : X > 0 \text{ or } \underline{\mathbb{E}}_{\mathcal{M}}(X) > 0\}. \tag{3}$$

Suppose that we have information about the relative importance of the i-th and the j-th criteria, i.e., the i-th criterion is more important than the j-th criterion with two positive parameters w_i and w_j. Let us return to the vector W_{ij} produced by the parameters w_i, w_j and consider again the relation $W_{ij} \succ 0_r$ (see Section 2). This relation can be written in the framework of desirable gambles as the condition $W_{ij} - 0_r \in \mathcal{D}$ or just $W_{ij} \in \mathcal{D}$. In other words, the information about the relative importance of the i-th and the j-th criteria can be represented as the condition that the vector W_{ij} belongs to the set of desirable gambles.

3.2 Maximality rule

Now we reformulate Noghin's theorem and prove it in terms of desirable gambles. It turns out that Nogin's solution coincides with Walley's maximality rule in imprecise probability theory. Let X and Y be two DA's. According to the maximality rule, we can state $X \succ Y$ when $\underline{\mathbb{E}}_{\mathcal{M}}(X - Y) > 0$, i.e., the difference $X - Y$ is a desirable gamble, $X - Y \in \mathcal{D}$. We will denote below the vector $Z = X - Y$ and its components $z_k = x_k - y_k$ for short.

Theorem 3.1: *Suppose that we have information about criteria in the form of preferences $W_{ij} \succ 0_r$ or $\Theta_{ij} \succ 0_r$. Then the preference $X \succ Y$ is valid if $X^* > Y^*$. Here $X^* = (x_1^*, ..., x_r^*)$ and $Y^* = (y_1^*, ..., y_r^*)$ such that*

$$x_j^* = \theta_{ji} x_i + (1 - \theta_{ji})\, x_j,\ x_k^* = x_k,\ k \neq j,$$
$$y_j^* = \theta_{ji} y_i + (1 - \theta_{ji})\, y_j,\ y_k^* = y_k,\ k \neq j.$$

Proof. Note that $X \succ Y$ if $X - Y = Z \in \mathcal{D}$ or $\mathbb{E}_P(Z) > 0$ for all $P \in \mathcal{M}$. The condition $\Theta_{ij} \succ 0_r$ restricts the set $\mathcal{M}$ of possible probability measures by the constraint $\mathbb{E}_P(W_{ij}) \geq 0$. If we denote $P = (\pi_1, ..., \pi_r)$, then the above constraint can be rewritten as $w_i \pi_i - w_j \pi_j \geq 0$ or $\theta_{ji} \pi_i - (1 - \theta_{ji})\, \pi_j \geq 0$. This implies that the set of all probability measures $\mathcal{M}$ is reduced to the subset $\mathcal{M}(ij) \subseteq \mathcal{M}$. The subset $\mathcal{M}(ij)$ is defined by the constraints

$$\sum_{k=1}^{r} \pi_k = 1,\ \pi_k \geq 0,\ \forall k \in N,\ \theta_{ji} \pi_i - (1 - \theta_{ji})\, \pi_j \geq 0.$$

Here $N = \{1, 2, ..., r\}$. Let us find extreme points of $\mathcal{M}(ij)$. They are

$$(0, ..., 0, 1_k, 0, ..., 0),\ \forall k \in N \backslash \{j\},$$

and

$$\pi_i = 1 - \theta_{ji},\ \pi_j = \theta_{ji},\ \pi_k = 0,\ k \in N \backslash \{j\},$$

Table 1 Values for the office location problem

	Closeness	Visibility	Image	Size	Comfort	Car parking
	C_1	C_2	C_3	C_4	C_5	C_6
A	100	60	100	75	0	90
B	20	80	10	30	100	30
C	80	70	0	0	10	100
D	70	50	30	55	30	90
E	40	60	90	100	60	70
F	0	0	70	0	80	0
G	60	100	20	50	50	80

or

$$\pi_i = \frac{w_j}{w_i + w_j},\ \pi_j = \frac{w_i}{w_i + w_j},\ \pi_k = 0,\ \forall k \in N \backslash \{j\}.$$

The last extreme point is produced by the equality $\theta_{ji}\pi_i - (1 - \theta_{ji})\,\pi_j = 0$. The extreme points define the set of probability distributions $\mathcal{M}(ij)$. Therefore, if we prove that the inequality $\mathbb{E}_P(Z) > 0$ is valid for extreme points, then this inequality will be valid for all $P \in \mathcal{M}(ij)$. The first $k-2$ extreme points give

$$\mathbb{E}_P(Z) = z_k,\ \forall k \in N \backslash \{i, j\}.$$

The last extreme point gives

$$\mathbb{E}_P(Z) = \pi_i z_i + \pi_j z_j = (1 - \theta_{ji})\, z_i + \theta_{ji} z_j.$$

At the same time, the condition $X^* > Y^*$ implies that $z_k > 0$ or $z_k = 0$ for all $k \neq j$, and $(1 - \theta_{ji})\, z_i + \theta_{ji} z_j > 0$. Hence $\mathbb{E}_P(Z) > 0$ for all $P \in \mathcal{M}(ij)$ and $X \succ Y$, as was to be proved. ■

Example 3.2: We consider an example of an office location problem provided by Goodwin and Wright in their book (Goodwin and Wright (2004)). Seven DA's are evaluated with respect to six criteria. The corresponding numerical values are shown in Table 1. It can be seen from Table 1 that all the DA's belong to the Pareto set. The DM is willing to pay $w_3 = 10$ units for the image in order to get $w_5 = 30$ units for the comfort. The provided information can be represented by the gamble $W_{53} = (0, 0, -10, 0, 30, 0) \in \mathcal{D}$ or equivalently by the gamble $\Theta_{53} = (0, 0, -0.25, 0, 0.75, 0) \in \mathcal{D}$. Here $\theta_{ji} = \theta_{35} = 0.75$. Then we write the modified Table 2 by using Noghin's theorem. One can see that there holds $B \succ F$. Hence, we reduce the Pareto set which now consists of six DA's A, B, C, D, E, G.

3.3 Interval dominance rule

Suppose we have the intervals of expectations $[\underline{\mathbb{E}}_{\mathcal{M}}(X), \overline{\mathbb{E}}_{\mathcal{M}}(X)]$ and $[\underline{\mathbb{E}}_{\mathcal{M}}(Y), \overline{\mathbb{E}}_{\mathcal{M}}(Y)]$ for X and Y, respectively. According to the interval dominance

Table 2 Modified values for the office location problem

	C_1	C_2	$0.75C_5 + (1-0.75)\,C_3$	C_4	C_5	C_6
A	100	60	25	75	0	90
B	20	80	77.5	30	100	30
C	80	70	7.5	0	10	100
D	70	50	30	55	30	90
E	40	60	67.5	100	60	70
F	0	0	77.5	0	80	0
G	60	100	42.5	50	50	80

rule, $X \succ Y$ when the interval $[\underline{\mathbb{E}}_{\mathcal{M}}(X), \overline{\mathbb{E}}_{\mathcal{M}}(X)]$ is completely on the right hand side of the interval $[\underline{\mathbb{E}}_{\mathcal{M}}(Y), \overline{\mathbb{E}}_{\mathcal{M}}(Y)]$, i.e., $\underline{\mathbb{E}}_{\mathcal{M}}(X) < \overline{\mathbb{E}}_{\mathcal{M}}(Y)$.

Let us introduce a set $\mathcal{I}$ of pairs of gambles which correspond to the interval dominance criterion. If a closed convex set of probability measures $\mathcal{M}$ is given, then we can define a set $\mathcal{I}$ as follows:

$$\mathcal{I} = \{(X,Y) : \mathbb{E}_P(X) - \mathbb{E}_Q(Y) > 0, \forall P, Q \in \mathcal{M}\}.$$

The above definition can be rewritten as

$$\mathcal{I} = \{(X,Y) : \underline{\mathbb{E}}_{\mathcal{M}}(X) - \overline{\mathbb{E}}_{\mathcal{M}}(Y) > 0\}.$$

Now we can say that the relation $X \succ Y$ is valid if and only if $X, Y \in \mathcal{I}$.

Note that the set $\mathcal{I}$ is reduced to the set $\mathcal{D}$ if $Y = 0_r$. If we again have information about the relative importance of the i-th and the j-th criteria and produce the vector W_{ij} by the parameters w_i, w_j, then the relation $W_{ij} \succ 0_r$ means that

$$\begin{aligned}\mathcal{I} &= \{(W_{ij}, 0_r) : \underline{\mathbb{E}}_{\mathcal{M}}(W_{ij}) - \overline{\mathbb{E}}_{\mathcal{M}}(0_r) > 0\} \\ &= \{(W_{ij}, 0_r) : \underline{\mathbb{E}}_{\mathcal{M}}(W_{ij}) > 0\}.\end{aligned}$$

It can be seen from the last expression that we get a set of desirable gambles $W_{ij} \in \mathcal{D}$ or $\Theta_{ij} \in \mathcal{D}$ which produce a set of probability measures $\mathcal{M}$.

The following theorem provides a simple computation procedure for reducing the Pareto set on the basis of the interval dominance rule.

Theorem 3.3: *Suppose that we have information about criteria in the form of preferences $W_{ij} \succ 0_r$ or $\Theta_{ij} \succ 0_r$. Then the preference $X \succ Y$ is valid if $x_k^* > y_l^*$ for all $k, l = 1, ...r$. Here $X^* = (x_1^*, ..., x_r^*)$ and $Y^* = (y_1^*, ..., y_r^*)$ such that*

$$\begin{aligned}x_j^* &= \theta_{ji} x_i + (1-\theta_{ji})\, x_j,\ x_k^* = x_k,\ k \neq j, \\ y_j^* &= \theta_{ji} y_i + (1-\theta_{ji})\, y_j,\ y_k^* = y_k,\ k \neq j.\end{aligned}$$

Proof. Note that $X \succ Y$ if $(X,Y) \in \mathcal{I}$ or $\underline{\mathbb{E}}_{\mathcal{M}}(X) - \overline{\mathbb{E}}_{\mathcal{M}}(Y) > 0$ for all $P \in \mathcal{M}$. The condition $W_{ij} \succ 0_r$ produces the set $\mathcal{M}(ij) \subseteq \mathcal{M}$ of probability measures with extreme points (see the proof of Theorem 3.1):

$$(0, ..., 0, 1_k, 0, ..., 0),\ \forall k \in N \backslash \{j\},$$

Table 3 Minimum and maximum values of DA's

	A	B	C	D	E	F	G
$\min X^*$	0	20	0	30	40	0	42.5
$\max X^*$	100	100	100	90	100	80	100

and

$$\pi_i = 1 - \theta_{ji},\ \pi_j = \theta_{ji},\ \pi_k = 0,\ k \in N \backslash \{j\}.$$

By using the first $r-1$ extreme points we can write the set of inequalities satisfying the condition $\underline{\mathbb{E}}_{\mathcal{M}}(X) - \overline{\mathbb{E}}_{\mathcal{M}}(Y) > 0$ as

$$x_k - y_l > 0,\quad \forall k, l \in N \backslash \{j\}.$$

The last extreme point jointly with one of the first $r-1$ extreme points produce three inequalities

$$\begin{aligned} x_k - (1-\theta_{ji})\, y_i - \theta_{ji} y_j &> 0, \quad \forall k \in N \backslash \{j\}, \\ (1-\theta_{ji})\, x_i + \theta_{ji} x_j - y_l &> 0, \quad \forall l \in N \backslash \{j\}, \\ (1-\theta_{ji})\, x_i + \theta_{ji} x_j - (1-\theta_{ji})\, y_i - \theta_{ji} y_j &> 0. \end{aligned}$$

All the above inequalities can be written in the compact form given in the theorem. ■

It is interesting to point out that Theorem 3.3 transforms the vectors X and Y in the same way as Theorem 3.1 to vectors X^* and Y^*, respectively. This is a very important feature. However, the ways for comparison of vectors X^* and Y^* are quite different. In order to apply the maximality rule, we compare r pairs of elements x_k^*, y_k^*, $k = 1, ..., r$. By applying the interval dominance rule, all r^2 pairs of elements x_k^*, y_l^*, $k, l = 1, ..., r$, are compared.

It should be noted that another way for comparison DA's is to find $\min X^* = \min\{x_1^*, ..., x_r^*\}$ and $\max X^* = \max\{x_1^*, ..., x_r^*\}$. Then the preference $X \succ Y$ is valid if $\min X^* > \max Y^*$. The above follows from the definition of the interval dominance criterion.

Example 3.4: Let us return to Example 3.2. It follows from Tables 2 and 3 that all DA's belong to the Pareto set which can not be reduced on the basis of the given judgment.

3.4 Interval bound dominance rule

The interval bound dominance rule, according to Destercke (2010), compares intervals of expectations $[\underline{\mathbb{E}}_{\mathcal{M}}(X), \overline{\mathbb{E}}_{\mathcal{M}}(X)]$ and $[\underline{\mathbb{E}}_{\mathcal{M}}(Y), \overline{\mathbb{E}}_{\mathcal{M}}(Y)]$ such that $X \succ Y$ when $\underline{\mathbb{E}}_{\mathcal{M}}(X) > \underline{\mathbb{E}}_{\mathcal{M}}(Y)$ and $\overline{\mathbb{E}}_{\mathcal{M}}(X) > \overline{\mathbb{E}}_{\mathcal{M}}(Y)$. The rule comes down to a pairwise comparison of the interval bounds. It also induces a partial order, i.e., a set of optimal decisions.

Let us introduce a set $\mathcal{B}$ of pairs of gambles which correspond to the interval dominance criterion. If a closed convex set of probability measures $\mathcal{M}$ is given, then we can define a set $\mathcal{B}$ as follows:

$$\mathcal{B} = \{(X,Y) : \underline{\mathbb{E}}_{\mathcal{M}}(X) - \underline{\mathbb{E}}_{\mathcal{M}}(Y) > 0,\ \overline{\mathbb{E}}_{\mathcal{M}}(X) - \overline{\mathbb{E}}_{\mathcal{M}}(Y) > 0\}.$$

Now we can say that the relation $X \succ Y$ is valid if and only if $X, Y \in \mathcal{B}$.

Theorem 3.5: *Suppose that we have information about criteria in the form of preferences $W_{ij} \succ 0_r$ or $\Theta_{ij} \succ 0_r$. Then the preference $X \succ Y$ is valid if $\min X^* > \min Y^*$, $\max X^* > \max Y^*$. Here $X^* = (x_1^*, ..., x_r^*)$ and $Y^* = (y_1^*, ..., y_r^*)$ such that*

$$x_j^* = \theta_{ji} x_i + (1 - \theta_{ji})\, x_j,\ x_k^* = x_k,\ k \neq j,$$
$$y_j^* = \theta_{ji} y_i + (1 - \theta_{ji})\, y_j,\ y_k^* = y_k,\ k \neq j,$$

where $\min X = \min\{x_1, ..., x_r\}$ *and* $\max X = \max\{x_1, ..., x_r\}$.

Proof. The proof is obvious if we consider the extreme points from the proof of Theorem 3.1. ■

Example 3.6: Let us return to Example 3.2. It follows from Table 3 that $B \succ A$, $B \succ C$, $E \succ D$, $E \succ F$, $G \succ E$, $G \succ B$. This implies that the Pareto set consists of one DA G.

The example shows that the maximality rule can be regarded as an intermediate solution the between interval dominance and interval bound dominance rules.

4 Sets of relative importance judgments

Now we consider cases when there are available a set of judgments about importance of criteria by means of pairs of positive parameters. Below we denote

$$M = \{i_1, ..., i_m\},\ L = \{j_1, ..., j_m\}.$$

Case 1. First we consider two judgments of the form: "The DM is willing to pay w_j units for the j-th criterion in order to get w_i units for the i-th criterion, and the DM is also willing to pay w_s units for the criterion with number s in order to get w_q units for the q-th criterion". Here we assume that $j \neq s$ and $i \neq q$. The above information can be formalized in the form of two preferences: $W_{ij} \succ 0_r$ and $W_{qs} \succ 0_r$.

In order to analyze different decision rules, we consider a set $\mathcal{M}$ of probability measures produced by two preferences $W_{ij} \succ 0_r$ and $W_{qs} \succ 0_r$. We will use the preferences $\Theta_{ij} \succ 0_r$ and $\Theta_{qs} \succ 0_r$ for short. It is obvious that the set $\mathcal{M}$ is the intersection of sets separately produced by every preference.

The set $\mathcal{M}$ is defined by the constraints

$$\sum_{k=1}^{r}\pi_k=1,\ \pi_k\geq 0,\ \forall k\in N,$$
$$\theta_{ji}\pi_i-(1-\theta_{ji})\,\pi_j\geq 0,\quad \theta_{sq}\pi_q-(1-\theta_{sq})\,\pi_s\geq 0,$$

Extreme points of $\mathcal{M}$ are

$$(0,...,0,1_k,0,...,0),\ \forall k\in N\backslash\{j\},$$

and

$$\pi_i=1-\theta_{ji},\ \pi_j=\theta_{ji},\ \pi_q=1-\theta_{sq},\ \pi_s=\theta_{sq},$$
$$\pi_k=0,\ \forall k\in N\backslash\{j,s\}.$$

Note that joint equations $\theta_{ji}\pi_i-\theta_{ij}\pi_j=0$, $\theta_{sq}\pi_q-\theta_{qs}\pi_s=0$ and $\pi_i+\pi_j+\pi_q+\pi_s=1$ do not give extreme points because we have three equations and four variables.

In the same way, we can construct the set $\mathcal{M}$ for arbitrary number of "non-intersecting" judgments. If we have m judgments of the form $\Theta_{i_1j_1}\succ 0_r,...,\Theta_{i_mj_m}\succ 0_r$, $i_k\neq i_l$ and $j_k\neq j_l$ by $k\neq l$, then the last extreme point is

$$\pi_i=1-\theta_{ji},\ \pi_j=\theta_{ji},\ j\in L,\ i\in M,$$
$$\pi_l=0,\ \forall l\in N\backslash(L\cup M).$$

Case 2. Let us consider another important case when $j=s$ and $i\neq q$, which considers two judgments of the form: "The DM is willing to pay w_j units for the j-th criterion in order to get w_i units for the i-th criterion, and the DM is also willing to pay w_s units for the criterion with number j in order to get w_q units for the q-th criterion". In this case, the set $\mathcal{M}$ is defined by the constraints

$$\theta_{ji}\pi_i-(1-\theta_{ji})\,\pi_j\geq 0,\quad \theta_{jq}\pi_q-(1-\theta_{jq})\,\pi_j\geq 0.$$

Separate equalities $\theta_{ji}\pi_i-\theta_{ij}\pi_j=0$ or $\theta_{jq}\pi_q-\theta_{qj}\pi_j=0$ do not produce extreme points because if $\pi_i>0$ and $\pi_j=1-\pi_i>0$, then, taking into account the equality $\pi_q=0$, the second inequality is not valid: $-\theta_{qj}\pi_j<0$ by $\theta_{qj}>0$ and $\pi_j>0$. So, we have to solve the following system of equations:

$$\begin{cases}\theta_{ji}\pi_i-(1-\theta_{ji})\,\pi_j=0,\\ \theta_{jq}\pi_q-(1-\theta_{jq})\,\pi_j=0,\\ \pi_i+\pi_j+\pi_q=1.\end{cases}$$

Its solution being the last extreme point is

$$\pi_i=\frac{\theta_{jq}\,(1-\theta_{ji})}{\theta_{ji}+\theta_{jq}-\theta_{ji}\theta_{jq}},\ \pi_j=\frac{\theta_{ji}\theta_{jq}}{\theta_{ji}+\theta_{jq}-\theta_{ji}\theta_{jq}},\ \pi_q=\frac{\theta_{ji}\,(1-\theta_{jq})}{\theta_{ji}+\theta_{jq}-\theta_{ji}\theta_{jq}}.$$

Let us divide every probability on π_j. Then we get

$$\pi_i=C\frac{1-\theta_{ji}}{\theta_{ji}},\ \pi_j=C,\ \pi_q=C\frac{1-\theta_{jq}}{\theta_{jq}}.$$

Here C is a normalizing coefficient such that the sum of probabilities is 1. The above gives us a simple way for representing the extreme point by m preferences of the form $\Theta_{i_1 j} \succ 0_r, ..., \Theta_{i_m j} \succ 0_r$. It is of the form:

$$\pi_{i_k} = C\frac{1-\theta_{ji}}{\theta_{ji}},\ \pi_j = C,\ i \in M,\ \pi_l = 0,\ l \in N\backslash(M \cup \{j\}).$$

Case 3. Finally, we consider a case when $j \neq s$ and $i = q$, for which we have two judgments of the form: "The DM is willing to pay w_j units for the j-th criterion in order to get w_i units for the i-th criterion, and the DM is also willing to pay w_s units for the criterion with number s in order to get w_q units for the i-th criterion". In this case, the set $\mathcal{M}$ is defined by the constraints

$$\theta_{ji}\pi_i - (1-\theta_{ji})\,\pi_j \geq 0, \quad \theta_{si}\pi_i - (1-\theta_{si})\,\pi_s \geq 0.$$

In the same way, we get the extreme point

$$\pi_i = \frac{(1-\theta_{si})\,(1-\theta_{ji})}{1-\theta_{ji}\theta_{si}},\ \pi_j = \frac{\theta_{ji}\,(1-\theta_{si})}{1-\theta_{ji}\theta_{si}},\ \pi_s = \frac{\theta_{si}\,(1-\theta_{ji})}{1-\theta_{ji}\theta_{si}}.$$

Let us divide every probability on π_i. Then we get

$$\pi_i = C,\ \pi_j = C\frac{\theta_{ji}}{1-\theta_{ji}},\ \pi_s = C\frac{\theta_{si}}{1-\theta_{si}}, \quad \pi_k = 0,\ k \in N\backslash\{i,j,s\}.$$

It can be seen that the extreme points for the last two cases are "symmetric" in some respect.

However, in contrast to Case 2, there are other extreme points when only one of the above inequalities is replaced by the corresponding equality. Then we have the following systems of equations:

$$\begin{cases} \theta_{ji}\pi_i - (1-\theta_{ji})\,\pi_j = 0, \\ \pi_i + \pi_j = 1, \end{cases} \qquad \begin{cases} \theta_{si}\pi_i - (1-\theta_{si})\,\pi_s = 0, \\ \pi_i + \pi_s = 1. \end{cases}$$

They provide the extreme points:

$$\pi_i = 1-\theta_{ji},\ \pi_j = \theta_{ji},\ \pi_k = 0,\ k \in N\backslash\{j\},$$
$$\pi_i = 1-\theta_{si},\ \pi_s = \theta_{si},\ \pi_k = 0,\ k \in N\backslash\{s\}.$$

If we divide every probability on π_i, then we get

$$\pi_i = C,\ \pi_j = C\frac{\theta_{ji}}{1-\theta_{ji}},\ \pi_k = 0,\ k \in N\backslash\{j\},$$
$$\pi_i = C,\ \pi_s = C\frac{\theta_{si}}{1-\theta_{si}},\ \pi_k = 0,\ k \in N\backslash\{s\}.$$

By comparing the last three extreme points, we can extend them on the general case of m preferences of the form $\Theta_{ij_1} \succ 0_r, ..., \Theta_{ij_m} \succ 0_r$. As a result, we get $\sum_{t=1}^{m}\binom{m}{t}$ extreme points of the form:

$$\pi_{j_k} = C\frac{\theta_{ji}}{1-\theta_{ji}},\ \pi_i = C,\ j \in L,\ \pi_l = 0,\ l \in N\backslash(L \cup \{i\}).$$

So, we have extreme points for all possible cases of the several simple judgments about relative importance of criteria. Now we can develop algorithms to reduce the Pareto optimal set for different decision rules.

Below we will study the above analyzed cases:

Case 1. $\Theta_{i_1j_1} \succ 0_r, ..., \Theta_{i_mj_m} \succ 0_r$ or $\Theta_{ij} \succ 0_r$, $\forall i \in M$, $\forall j \in L$.

Case 2. $\Theta_{i_1j} \succ 0_r, ..., \Theta_{i_mj} \succ 0_r$ or $\Theta_{ij} \succ 0_r$, $\forall i \in M$.

Case 3. $\Theta_{ij_1} \succ 0_r, ..., \Theta_{ij_m} \succ 0_r$ or $\Theta_{ij} \succ 0_r$, $\forall j \in L$.

Moreover, we suppose that $\varnothing \subseteq M$ and $\varnothing \subseteq L$. We also denote

$$R(Q,X) = C(Q)\left(x_i + \sum_{k\in Q}\frac{\theta_{ki}}{1-\theta_{ki}}x_k\right),$$

where $C(Q)$ is the normalized coefficient defined as

$$C(Q) = \left(1 + \sum_{k\in Q}\frac{\theta_{ki}}{1-\theta_{ki}}\right)^{-1}.$$

The above expressions can be rewritten in terms of parameters w_i as follows:

$$R(Q,X) = C(Q)\left(x_i + \sum_{k\in Q}\frac{w_i}{w_k}x_k\right),$$
$$C(Q) = \left(1 + \sum_{k\in Q}\frac{w_i}{w_k}\right)^{-1}.$$

4.1 Maximality rule

Theorem 4.1: *The preference $X \succ Y$ is valid if $X^* > Y^*$. Here $X^* = (x_1^*, ..., x_r^*)$ and $Y^* = (y_1^*, ..., y_r^*)$ such that: Case 1.*

$$x_j^* = (1-\theta_{ji})\,x_i + \theta_{ji}x_j,\ i \in M,\ j \in L,$$
$$y_j^* = (1-\theta_{ji})\,y_i + \theta_{ji}y_j,\ i \in M,\ j \in L,$$
$$x_l^* = x_l,\ y_l^* = y_l,\ l \in N\backslash(L\cup M).$$

Case 2.

$$x_j^* = x_j + \sum_{i\in M}\frac{1-\theta_{ji}}{\theta_{ji}}x_i,\ y_j^* = y_j + \sum_{i\in M}\frac{1-\theta_{ji}}{\theta_{ji}}y_i,$$
$$x_l^* = x_l,\ y_l^* = y_l,\ l \in N\backslash\{j\}.$$

Case 3. The preference $X \succ Y$ is valid if there holds the following inequality:

$$\min_{Q\subseteq M} R(Q, X-Y) > 0.$$

Proof. The proof directly follows from the extreme points. Cases 1 and 2 are can be derived in the same way as in the proof of Theorem 3.1. There are a lot of extreme points in Case 3, which can not be represented in a simple way. Therefore, a direct enumeration of all the points is carried out. ■

Cases 1 and 2 of Theorem 4.1 are very similar to Noghin's results. However, they have been derived in a quite different way by using the imprecise probability theory.

The proofs of the following theorems are obvious.

4.2 Interval dominance rule

Theorem 4.2: *The preference $X \succ Y$ is valid if $x_k^* > y_l^*$ for all $k, l = 1, ...r$. Here $X^* = (x_1^*, ..., x_r^*)$ and $Y^* = (y_1^*, ..., y_r^*)$ such that Case 1.*

$$x_j^* = (1 - \theta_{ji})\, x_i + \theta_{ji} x_j,\ i \in M,\ j \in L,$$
$$y_j^* = (1 - \theta_{ji})\, y_i + \theta_{ji} y_j,\ i \in M,\ j \in L,$$
$$x_l^* = x_l,\ y_l^* = y_l,\ l \in N \backslash (L \cup M).$$

Case 2.

$$x_j^* = x_j + \sum_{i \in M} \frac{1 - \theta_{ji}}{\theta_{ji}} x_i,\ y_j^* = y_j + \sum_{i \in M} \frac{1 - \theta_{ji}}{\theta_{ji}} y_i,$$
$$x_l^* = x_l,\ y_l^* = y_l,\ l \in N \backslash \{j\}.$$

Case 3. The preference $X \succ Y$ is valid if the following inequality is valid:

$$\min_{Q \subseteq M} R(Q, X) > \max_{Q \subseteq M} R(Q, Y),$$

4.3 Interval bound dominance rule

Theorem 4.3: *The preference $X \succ Y$ is valid if $\min X^* > \min Y^*$, $\max X^* > \max Y^*$. Here $\min X$ and $\max X$ are defined in Theorem 3.5. $X^* = (x_1^*, ..., x_r^*)$ and $Y^* = (y_1^*, ..., y_r^*)$ are such that Case 1.*

$$x_j^* = (1 - \theta_{ji})\, x_i + \theta_{ji} x_j,\ i \in M,\ j \in L,$$
$$y_j^* = (1 - \theta_{ji})\, y_i + \theta_{ji} y_j,\ i \in M,\ j \in L,$$
$$x_l^* = x_l,\ y_l^* = y_l,\ l \in N \backslash (L \cup M).$$

Case 2.

$$x_j^* = x_j + \sum_{i \in M} \frac{1 - \theta_{ji}}{\theta_{ji}} x_i,\ y_j^* = y_j + \sum_{i \in M} \frac{1 - \theta_{ji}}{\theta_{ji}} y_i,$$
$$x_l^* = x_l,\ y_l^* = y_l,\ l \in N \backslash \{j\}.$$

Case 3. The preference $X \succ Y$ is valid if the following two inequalities are valid:

$$\min_{Q \subseteq M} R(Q, X) > \min_{Q \subseteq M} R(Q, Y),\quad \max_{Q \subseteq M} R(Q, X) > \max_{Q \subseteq M} R(Q, Y).$$

Table 4 Minimum and maximum values of DA's

	A	B	C	D	E	F	G
$\min X^*$	68	20	40	54	40	0	55
$\max X^*$	100	68	80	70	70	0	92

Example 4.4: We return to Example 3.2. The following information is available now: "The DM is willing to pay $w_3 = 10$ units for the image in order to get $w_5 = 30$ units for the comfort." "The DM is willing to pay $w_4 = 20$ units for the size in order to get $w_1 = 20$ units for the closeness." "The DM is willing to pay $w_2 = 10$ units for the visibility in order to get $w_1 = 40$ units for the closeness." The provided information can be represented by the gambles

$$\Theta_{53} = (0, 0, -0.25, 0, 0.75, 0),$$
$$\Theta_{14} = (0.5, 0, 0, -0.5, 0, 0),$$
$$\Theta_{12} = (0.8, -0.2, 0, 0, 0, 0),$$

where $\theta_{35} = 0.75$, $\theta_{41} = 0.5$ and $\theta_{21} = 0.8$. The first judgment has been used in Example 3.2. The second and the third judgments correspond to Case 3 with $M = \{4, 2\}$. Then Q can be $\varnothing$, $\{4\}$, $\{2\}$, $\{4, 2\}$. Hence

$$R(\varnothing, X) = x_1,\ R(\{4\}, X) = C(\{4\}) \left(x_1 + \frac{\theta_{41}}{1 - \theta_{41}} x_4 \right),$$
$$R(\{2\}, X) = C(\{2\}) \left(x_1 + \frac{\theta_{21}}{1 - \theta_{21}} x_2 \right),$$
$$R(\{4, 2\}, X) = C(\{4, 2\}) \left(x_1 + \frac{\theta_{41}}{1 - \theta_{41}} x_4 + \frac{\theta_{21}}{1 - \theta_{21}} x_2 \right),$$

where

$$C(\{4\}) = 1 - \theta_{41} = 0.5,\ C(\{2\}) = 1 - \theta_{21} = 0.2,$$
$$C(\{4, 2\}) = \frac{(1 - \theta_{21})(1 - \theta_{41})}{1 - \theta_{21}\theta_{41}} = 0.167.$$

By applying the maximality rule and taking into account the above three judgments, we get the following preferences: $A \succ D$ ($\min_{Q \subseteq M} R(Q, A - D) = 14$), $A \succ E$ ($\min_{Q \subseteq M} R(Q, A - E) = 5.8$), $G \succ B$ ($\min_{Q \subseteq M} R(Q, G - B) = 23.38$). In other words, the Pareto optimal set consists of three DA's: A, C, G. If we use the interval dominance rule, then the DA F can be removed (see Table 4). Hence, we reduce the Pareto set which now consists of six DA's A, B, C, D, E, G. In case of the interval bound dominance rule, the Pareto optimal set consists of one DA A.

5 Conclusion

A method for solving a MCDM problem with the elicited information about criteria of a special form has been proposed in the paper. The main feature of the method is that it is based on reducing a set of Pareto optimal solutions and does not use aggregation of criteria for solving the problem. The additional information applied in the proposed method is rather natural because DM's or experts are usually able to provide parameters w_i and w_j whose simple behavior interpretation is considered in Section 2 and in numerical examples.

It has been shown in the paper that Noghin's theory of relative importance of criteria can be easily represented in terms of sets of desirable gambles and many statements of the theory can be proved by means of imprecise probability theory.

It should be noted that the main decision rules used in imprecise probability theory, including maximality, interval dominance and interval bound dominance rules, have been studied in the paper and have been applied to the considered MCDM problems. However, there exist other decision rules exploited under partial information about criteria, which have not been analyzed here, for instance, Gamma-maximinity and E-admissibility Walley (1991). The corresponding methods for solving the MCDM problems on the basis of these decision rules are a topic for further research.

Another open question is a strong recommendation for selecting a specific decision rule. Every rule analyzed can be successfully applied to some problems and can provide unsatisfactory solutions for other applied problems. Possible recommendations are very important.

One could see from the proposed definitions and theorems that all the mathematical expressions are rather simple from the computation point of view. They do not require special procedures, for example, linear programming, for reducing the set of Pareto solutions. Moreover, one could see from the paper that the key concept used for getting simple mathematical expressions is the set of extreme points of a convex set of probability distributions. The set of extreme points is a powerful tool to avoid solving linear programming problems.

It should be noted that only some types of sets of comparative judgments has been studied in the paper, and simple computation procedures have been derived only for these sets of judgments. However, arbitrary sets of the considered judgments can be processed by means of linear programming problem. Moreover, the MCDM problems can be extended on a more general case of groups of experts or DM's. In this case, we get second-order models (Utkin (2003)), which are a basis for further work.

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