Fuzzy decision making using the imprecise Dirichlet model

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Abstract: In most applications, probabilities of states of nature in decision making are not known exactly due to a lack of complete information. If the available information is represented by a small number of statistical data, Walley's imprecise Dirichlet model may be regarded as a tool for determining interval probabilities of states of nature. It turns out that the resulting expected utilities constitute fuzzy sets and the initial decision problem is reduced to a fuzzy decision problem. A numerical example illustrates the proposed approach to solving decision problems under the scarce information about states of nature. The approach is compared with ?-contaminated (robust) models. This approach can also be applied to construction of fuzzy sets on the basis of statistical observations.

Keywords: decision problem; fuzzy sets; robust model; imprecise Dirichlet model; expected utilities.

Reference to this paper should be made as follows: Utkin, L.V. and Zhuk, Y.A. (xxxx) 'Imprecise voting multi-criteria decision problems for solving the logistics and transportation management problems', *Int. J. Mathematics in Operational Research*, Vol. x, No. x, pp.xxx–xxx.

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1 Introduction

Consider the basic model of decision theory: One has to choose an *action* from a non-empty, finite set $\mathbb{A} = \{a_1, ..., a_n\}$ of possible actions. The consequences of every action depend on the true, but unknown *state of nature* $\vartheta \in \Theta = \{\vartheta_1, ..., \vartheta_m\}$. The corresponding outcome is evaluated by the *utility function*

$$u \; : \; \begin{array}{l} (\mathbb{A} \times \Theta) \to \mathbb{R} \\ (a, \vartheta) \mapsto u(a, \vartheta) \end{array},$$

and by the associated random variable $\mathbf{u}(\mathbf{a})$ on $(\Theta, \mathcal{P}o(\Theta))$ taking the values $u(a, \vartheta)$. Alternatively a loss function $l(a, \vartheta)$ is assigned, which can be embedded into the framework proposed by setting $u(a, \vartheta) = -l(a, \vartheta)$.

This model contains the essentials of every (formalized) decision situation under uncertainty and is applied in a huge variety of disciplines. If the states of nature are produced by a perfect random mechanism (e.g. an ideal lottery), and the corresponding probability measure $\pi(\cdot)$ on $((\Theta, \mathcal{P}o(\Theta)))$ is completely known, the Bernoulli principle is nearly unanimously favored. One chooses that action a_k which maximizes the expected utility $\mathbb{E}_\pi \mathbf{u}_k := \sum_{j=1}^{m} (u(a_k, \vartheta_j) \cdot \pi(\vartheta_j))$. For simplicity, we will denote $\pi_j = \pi(\vartheta_j)$ and $u_{rj} = u(a_r, \vartheta_j)$.

In most practical applications, however, the true state of nature can not be understood as arising from an ideal random mechanism. And even if so, the corresponding probability distribution will be not known exactly. Obviously, decision making model in this case depends on initial information about states of nature (more precisely about the probability distribution of states of nature). An efficient approach for solving the decision problem under initial data in the form of lower and upper probabilities (expectations) in the framework of imprecise probability theory (Kuznetsov (1991), Walley (1991), Weichselberger (2001)) has been proposed by Augustin (2001, 2002). A similar approach for solving the decision problem taking into account the second-order probabilities (Utkin (2003)) has been discussed by Utkin and Augustin (2003). However, the available information about states of nature may be represented in the form of numbers of statistical observations of states. In this case, the proposed models do not work because the total number of observations may be very small and it is very difficult to infer statistical characteristics describing states of nature. One of the most appropriate approaches in this case is to use the Dirichlet model. The Dirichlet model has been widely adopted to many applications due to interesting properties of this model, in particular, due to the important fact that Dirichlet density functions constitute a conjugate family of density functions (Robert (1994)). A very promising generalization of existent Dirichlet models is Walley's imprecise Dirichlet model (Walley (1996a)) taking into account incompleteness of statistical data.

We show in this paper how to reduce the initial decision problem to a fuzzy decision problem by a small number of statistical observations of states of nature. It is carried out by constructing fuzzy expected utilities of actions on the basis of a set of the imprecise Dirichlet models. This set of models is produced by a hyperparameter of the Dirichlet model.

Throughout the paper we assume the precise utilities are given.

Suppose that there is available a set $(n_1, ..., n_m)$ of statistical observations (measurements, expert judgments) of states of nature ϑ. Our aim is to construct a reasonable criterion of decision making and to find the optimal action in accordance with this criterion.

2 Background for the research

A comprehensive critical analysis of the role of fuzzy sets in decision making was proposed by Dubois (2011). The author discusses membership functions, aggregation operations, linguistic variables, fuzzy intervals and the valued preference relations. Dubois (2011) argues that if a precise value in the real line provided by an expert is often ill-known, it can be more faithfully represented by an interval or a fuzzy interval. A review of fuzzy ranking procedures is also provided in the paper (Dubois (2011)).

Several new methods of applying the fuzzy sets to decision problems were proposed also in other works. In particular, Yeh and Chang (2009) presents a new fuzzy multi-criteria decision making approach for evaluating decision alternatives involving subjective judgements made by a group of decision makers. The paper (Ye (2010)) exploits fuzzy sets for situations where the information about criteria weights for alternatives is completely unknown. Kasperski and Zielinski (2010) studies an optimization problem in which closed intervals and fuzzy intervals model uncertain element weights. The notion of a deviation interval is introduced, which allows us to characterize the optimality and the robustness of solutions and elements.

Partial information in the form of intervals which can be regarded as a special case of fuzzy sets was investigated by Pal et al. (2011). Guo and Tanaka (2010) used interval probabilities when it is difficult to characterize the uncertainty by point-valued probabilities due to the partially known information. The authors also studied interval probabilities, the interval entropy, interval expectations, interval variances, interval moments, and the decision criteria with interval probabilities. The results provided in this work show that the proposed methods provide a novel and effective alternative for decision making when point-valued subjective probabilities are inapplicable due to the partially known information. An approach on the basis of rough sets in decision problems was provided by Dembczynski et al. (2009). According to this approach, the imprecisions are given in the form of intervals of possible values for which second-order rough approximations are introduced.

Decision making procedures for cases when the information about states of nature is represented by sets of probability measures were considered by Destercke (2010). Similar procedures were proposed by Utkin and Augustin (2005, 2007) and Troffaes (2007).

A lot of methods for constructing membership functions of fuzzy sets can be found in literature. One of the approaches for constructing membership functions when we have statistical data was provided by Nasibov and Peker (2011). According to this approach, a frequency tables of a data set are used in the determination of membership functions. Parameters of the so-called exponential membership function are generated via a minimization problem. In another work of the same authors (Nasibov and Peker (2008)), a parametric fuzzy approximation method is offered. It is based on the decision maker's strategy as an extension of the trapezoidal approximation of a fuzzy number. An interesting approach for finding the membership function of a fuzzy number is provided by Chutia et al. (2010). The authors show that the Dubois-Prade left and right reference functions of a fuzzy number viewing as a distribution function and a complementary distribution function, respectively, lead to a very simple alternative method of finding the membership function of a fuzzy number. A method for constructing membership functions taking into account statistical data was proposed by Cai (1993).

There are several interesting and relevant papers related to decision making by different forms of initial data uncertainty such as (Borgonovo (2009); Ferhatosmanoglu et al. (2009); Hasuike and Ishii (2009); Kumar and Kaur (2011); Jha et al. (2011); Xidonas et al. (2010)).

3 Walley's imprecise Dirichlet model

The *Dirichlet* (s,α) *prior distribution* for π, where $\alpha = (\alpha_1, ..., \alpha_m)$, has probability density function (DeGroot (1970); Wilks (1962))

$$p(\pi) = C(s,\alpha) \cdot \prod_{j=1}^{m} \pi_j^{s\alpha_j - 1},$$

where $s > 0$, $0 < \alpha_j < 1$ for $j = 1, ..., m$, $\alpha \in S(1,m)$, and the proportionality constant C is determined by the fact that the integral of $p(\pi)$ over the simplex of possible values of π is 1 and

$$C(s,\alpha) = \Gamma(s) \left(\prod_{j=1}^{m} \Gamma(s\alpha_j) \right)^{-1}.$$

Here α_i is the mean of π_i under the Dirichlet prior and s determines the influence of the prior distribution on posterior probabilities. $\Gamma(\cdot)$ is the Gamma-function which satisfies $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(1) = 1$. $S(1,m)$ denotes the interior of the m-dimensional unit simplex.

Walley (1996a) pointed out several reasons for using a set of Dirichlet distributions to model prior ignorance about probabilities π:

1. Dirichlet prior distributions are mathematically tractable because they generate Dirichlet posterior distributions;

2. sets of Dirichlet distributions are very rich, because they produce the same inferences as their convex hulls and any prior distribution can be approximated by a finite mixture of Dirichlet distributions;

3. the most common Bayesian models for prior ignorance about probabilities π are Dirichlet distributions.

The *imprecise Dirichlet model* is defined by Walley (1996a) as the set of all Dirichlet (s, α) distributions such that $\alpha \in S(1, m)$.

For the imprecise Dirichlet model, the *hyperparameter* s determines how quickly upper and lower probabilities of events converge as statistical data accumulate. Walley (1996a) defined s as a number of observations needed to reduce the imprecision (difference between upper and lower probabilities) to half its initial value. Smaller values of s produce faster convergence and stronger conclusions, whereas large values of s produce more cautious inferences. At the same time, the value of s must not depend on m or a number of observations. The detailed discussion concerning the parameter s and the imprecise Dirichlet model can be found in papers Bernard (2002); Coolen (1997); Smithson et al. (1999); Walley (1996a). The application of the model in reliability was studied by Coolen (1997). This model was also applied to the game theory for choosing a strategy in a two-player game by Quaeghebeur and de Cooman (2003). An approach to dealing with incomplete sets of multivariate categorical data by exploiting Walley's imprecise Dirichlet model was studied by Zaffalon (2002).

Let A be any non-trivial subset of a sample space $\{\vartheta_1, ..., \vartheta_m\}$, and let $n(A)$ denote the observed number of occurrences of A in the N trials, $n(A) = \sum_{\vartheta_j \in A} n_j$. Then, according to Walley (1996a), the predictive probability $P(A, s)$ under the Dirichlet posterior distribution is

$$P(A, s) = \frac{n(A) + s\alpha(A)}{N + s},$$

where $\alpha(A) = \sum_{\vartheta_j \in A} \alpha_j$.

By maximizing and minimizing α_j under restriction $\alpha \in S(1, m)$, we obtain the posterior upper and lower probabilities of A:

$$\underline{P}(A, s) = \frac{n(A)}{N + s}, \ \overline{P}(A, s) = \frac{n(A) + s}{N + s}.$$

4 The expected utility by the imprecise Dirichlet model

Let us consider an approach for decision making under condition that π has the precise Dirichlet distribution. In this case, the expected utility $\mathbb{E}_\pi \mathbf{u}_r$ can be regarded as a conditional expectation. Denote the dependency of $\mathbb{E}_\pi \mathbf{u}_r$ on the parameter s by $\mathbb{E}_\pi^{(s)} \mathbf{u}_r$. Then, by using the Bayesian model, we find the unconditional expected utility $\mathbb{E}^{(s)} \mathbf{u}_r$ as

$$\mathbb{E}^{(s)} \mathbf{u}_r = \int_\Theta \sum_{i=1}^m u_{ri} \pi_i p(\pi) \mathrm{d}\pi$$

$$= \sum_{i=1}^{m} u_{ri} \int_{\Theta} \pi_i C(s,\alpha) \prod_{j=1}^{m} \pi_j^{s\alpha_j - 1} \mathrm{d}\pi, \ \Theta = S(1,m).$$

The integral in the above expression is nothing but the expectation $\mathbb{E}_p^{(s)}\pi_i$ of π_i with respect to $p(\pi)$. This implies that

$$\mathbb{E}^{(s)}\mathbf{u}_r = \sum_{i=1}^{m} u_{ri} \cdot \mathbb{E}_p^{(s)}\pi_i.$$

According to Walley (1996a), there holds

$$\mathbb{E}_p^{(s)}\pi_i = (n_i + s\alpha_i)/(N+s),$$

where n_i denotes the number of observations of the i-th state of nature in N trials.

Then

$$\mathbb{E}^{(s)}\mathbf{u}_r = \sum_{i=1}^{m} u_{ri} \cdot \mathbb{E}_p^{(s)}\pi_i = \sum_{i=1}^{m} u_{ri} \cdot \frac{n_i + s\alpha_i}{N+s}.$$

By using the imprecise Dirichlet model, we can write the lower and upper expected utilities as the following optimization problems:

$$\underline{\mathbb{E}}^{(s)}\mathbf{u}_r = \inf_{\alpha\in S(1,m)} \mathbb{E}^{(s)}\mathbf{u}_r, \ \overline{\mathbb{E}}^{(s)}\mathbf{u}_r = \sup_{\alpha\in S(1,m)} \mathbb{E}^{(s)}\mathbf{u}_r. \tag{1}$$

In other words, we have to solve two linear programming problems in order to find $\underline{\mathbb{E}}^{(s)}\mathbf{u}_r$ and $\overline{\mathbb{E}}^{(s)}\mathbf{u}_r$. Let us rewrite $\mathbb{E}^{(s)}\mathbf{u}_r$ as follows:

$$\mathbb{E}^{(s)}\mathbf{u}_r = \frac{1}{N+s}\sum_{i=1}^{m} u_{ri} n_i + \frac{s}{N+s}\sum_{i=1}^{m} u_{ri}\alpha_i.$$

Since extreme points of the simplex $S(1,m)$ are $(1,0,..,0)$, $(0,1,..,0)$,..., $(0,0,..,1)$, then the optimal solutions can be found by substituting these points into objective functions (1). As a result, we have

$$\underline{\mathbb{E}}^{(s)}\mathbf{u}_r = \frac{1}{N+s}\sum_{i=1}^{m} u_{ri} n_i + \frac{s}{N+s}\min_{i=1,\dots,m} u_{ri}, \tag{2}$$

$$\overline{\mathbb{E}}^{(s)}\mathbf{u}_r = \frac{1}{N+s}\sum_{i=1}^{m} u_{ri} n_i + \frac{s}{N+s}\max_{i=1,\dots,m} u_{ri}. \tag{3}$$

In particular, if $N=0$, then there hold $\underline{\mathbb{E}}^{(s)}\mathbf{u}_r = \min_{i=1,\dots,m} u_{ri}$, $\overline{\mathbb{E}}^{(s)}\mathbf{u}_r = \max_{i=1,\dots,m} u_{ri}$. In this case, lower and upper expected utilities do not depend on s. If $N\to\infty$, $n_i = \pi_i N$, $i \le m$, then $\underline{\mathbb{E}}^{(s)}\mathbf{u}_r = \overline{\mathbb{E}}^{(s)}\mathbf{u}_r = \sum_{i=1}^{m} u_{ri}\pi_i$.

The above expressions have been proposed by Utkin and Augustin (2005, 2007). It is worth noticing that we have obtained intervals of expected utilities whose bounds $\underline{\mathbb{E}}^{(s)}\mathbf{u}_r$ and $\overline{\mathbb{E}}^{(s)}\mathbf{u}_r$ depend on the parameter s. In spite of various rules (Walley (1996a)) for choosing s, it is difficult to assign a certain value for s in many cases. Moreover, the sense of s is not so clear for practitioners and different values of s lead to different optimal actions. Therefore, it is necessary to develop a more convenient way for decision making.

5 Interval expected utilities as fuzzy sets

Note that intervals $[\underline{\mathbb{E}}^{(s)}\mathbf{u}_r, \overline{\mathbb{E}}^{(s)}\mathbf{u}_r]$ by different values of s produce a set of nested intervals. At that, if $s \to 0$, then $\underline{\mathbb{E}}^{(0)}\mathbf{u}_r = \overline{\mathbb{E}}^{(0)}\mathbf{u}_r$. If $s \to \infty$, then $\underline{\mathbb{E}}^{(s)}\mathbf{u}_r = \min_{i=1,...,m} u_{ri}$, $\overline{\mathbb{E}}^{(s)}\mathbf{u}_r = \max_{i=1,...,m} u_{ri}$. Let us introduce a function $\mu(s)$ with the following properties: $\mu(0) = 1$, $\mu(\infty) = 0$, and $\mu(s)$ is non-increasing with s. Examples of such functions are $\mu(s) = \exp(-s)$, $\mu(s) = (1+s)^{-1}$. Thus, we have a set of nested intervals characterized by real numbers $\mu \in [0,1]$. This set can be viewed as a fuzzy set $\widetilde{\mathbb{E}}\mathbf{u}_r$ of the expected utility with the membership function μ.

Now we can write a new criterion of decision making. An action a_k is optimal iff for all $r = 1, ..., n$, $\widetilde{\mathbb{E}}\mathbf{u}_k \geq \widetilde{\mathbb{E}}\mathbf{u}_r$.

It can be seen from the above reasoning that we get the fuzzy decision problem. The next question is how to order or to compare the fuzzy sets? This is one of the most controversial matters in fuzzy literature. Most ranking methods are based on transforming a fuzzy set into a real number called by the *ranking index*. Here we use the index proposed by Campos and Gonzalez (1989), which can be written in terms of the considered decision problem as

$$F_r = \int_0^1 \left(\eta \underline{\mathbb{E}}^{(\mu)}\mathbf{u}_r + (1-\eta)\overline{\mathbb{E}}^{(\mu)}\mathbf{u}_r \right) \mathrm{d}\mu. \tag{4}$$

Here $\eta \in [0,1]$ is a parameter of pessimism. Then we write a new criterion of decision making. An action a_k is optimal iff for all $r = 1, ..., n$, there holds

$$F_k \geq F_r.$$

It is worth noticing that the used ranking index F_r can be interpreted in another way. The branches of the membership function produce a set of probability distributions (Dubois and Prade (1988); Walley (1996b)) with lower $\underline{H}(x)$ and upper $\overline{H}(x)$ survival functions of a random variable X determined as

$$\underline{H}(x) = \begin{cases} 1-\mu, & x = \underline{\mathbb{E}}^{(\mu)}\mathbf{u}_r \\ 0, & x = \overline{\mathbb{E}}^{(\mu)}\mathbf{u}_r \end{cases},$$

$$\overline{H}(x) = \begin{cases} 1, & x = \underline{\mathbb{E}}^{(\mu)}\mathbf{u}_r \\ \mu, & x = \overline{\mathbb{E}}^{(\mu)}\mathbf{u}_r \end{cases}.$$

For the given lower and upper survival functions, lower $\underline{\mathbb{E}}X$ and upper $\overline{\mathbb{E}}X$ expectations can be found as follows:

$$\underline{\mathbb{E}}X = \int_{-\infty}^{\infty} \underline{H}(x)\mathrm{d}x, \quad \overline{\mathbb{E}}X = \int_{-\infty}^{\infty} \overline{H}(x)\mathrm{d}x.$$

But these expectations can be represented as the following integrals:

$$\underline{\mathbb{E}}X = \int_0^1 \underline{\mathbb{E}}^{(\mu)}\mathbf{u}_r \mathrm{d}\mu, \quad \overline{\mathbb{E}}X = \int_0^1 \overline{\mathbb{E}}^{(\mu)}\mathbf{u}_r \mathrm{d}\mu.$$

This implies that $F_r = \eta\underline{\mathbb{E}}X + (1-\eta)\overline{\mathbb{E}}X$. From this point of view, the parameter η can be regarded as a *caution parameter* (Schubert (1995); Weichselberger (2001)) in decision making and the decision problem can be regarded as a problem with the imprecise information about states of nature (Augustin (2002)). The caution parameter reflects the degree of ambiguity aversion. The more ambiguity averse the decision maker is, the higher is the influence of the lower interval limit of generalized expected utility. $\eta = 1$ corresponds to strict ambiguity aversion, $\eta = 0$ expresses maximal ambiguity seeking attitudes. Thus, we can observe a very interesting link between the parameter of pessimism for the considered ranking index and the caution parameter in decision making.

Let us try to simplify (4). Denote

$$\Psi(s) = \eta\underline{\mathbb{E}}^{(s)}\mathbf{u}_r + (1-\eta)\overline{\mathbb{E}}^{(s)}\mathbf{u}_r.$$

Then

$$F_r = -\int_0^\infty \Psi(s)\mathrm{d}\mu(s).$$

By integrating by parts, we obtain

$$\begin{aligned} F_r &= \Psi(0)\mu(0) - \Psi(\infty)\mu(\infty) + \int_0^\infty \mu(s)\mathrm{d}\Psi(s) \\ &= \Psi(0) + \int_0^\infty \mu(s)\mathrm{d}\Psi(s). \end{aligned}$$

After substituting (2)-(3) into the above expression and simplifying it, we get

$$F_r = A_r + ND_r \int_0^\infty \mu(s)(N+s)^{-2}\mathrm{d}s, \tag{5}$$

where

$$D_r = -A_r + G_r,\ A_r = N^{-1}\sum_{i=1}^m u_{ri}n_i,$$

$$G_r = \eta \min_{i=1,\dots,m} u_{ri} + (1-\eta)\max_{i=1,\dots,m} u_{ri}.$$

Here A_r is the expected utility of the r-th action under the given observations $n_1,\dots,n_m$; G_r is the expected utility in accordance with Hurwicz criterion with optimism parameter $1-\eta$; D_r can be regarded as some correction for G_r taking into account available statistical data.

By using the mean value of an integral, we rewrite (5) as

$$F_r = A_r + ND_r\mu(\gamma)\int_0^\infty (N+s)^{-2}\mathrm{d}s = A_r + D_r\mu(\gamma),$$

where $\gamma \in [0, \infty)$.

Hence

$$F_r = A_r \cdot (1 - \mu(\gamma)) + G_r \cdot \mu(\gamma).$$

It follows from the above expression that the index F_r is a convex combination of the Hurwicz criterion and the expected utility.

6 Fuzzy neigbourhoods

The probabilities of states of nature also can be considered in the framework of *ε-contaminated (robust) models* (Huber (1981)). They are constructed by eliciting a Bayesian prior distribution P as an estimate of the true prior distribution. By using the notation introduced in the previous sections, we formulate the ε-contaminated model as follows. This is a class of probabilities which for fixed $\varepsilon \in (0,1)$ and π_i is the set $\mathcal{M}(\varepsilon) = \{(1-\varepsilon)\pi_i + \varepsilon q_i\}$, where q_i is arbitrary and $q_1 + ... + q_m = 1$. Since the neighbourhoods of this type are nested, i.e., $\mathcal{M}(\varepsilon_1) \subseteq \mathcal{M}(\varepsilon_2)$ whenever $\varepsilon_1 \leq \varepsilon_2$, then Walley (1997) proposed the so-called fuzzy neighbourhoods or fuzzy contaminated models. According to these models, for $0 \leq \varepsilon \leq 0$, $\mathcal{M}(\varepsilon)$ is the set of all probabilities with the lower bound $(1-\varepsilon\}\pi_i$ and the upper bound $(1-\varepsilon\}\pi_i + \varepsilon$.

Let us find the lower and upper expected utilities generated by the set $\mathcal{M}(\varepsilon)$. For a fixed ε and q_i, the expected utility is of the form:

$$\begin{aligned}
\mathbb{E}^{(\varepsilon)}\mathbf{u}_r &= \sum_{i=1}^{m} u_{ri}\left((1-\varepsilon)\pi_i + \varepsilon q_i\right) \\
&= (1-\varepsilon)\sum_{i=1}^{m} u_{ri}\pi_i + \varepsilon \sum_{i=1}^{m} u_{ri} q_i.
\end{aligned}$$

By minimizing and maximizing the above expected utility over all possible probability distributions $(q_1, ..., q_m)$ and taking into account the fact that optimal solutions can be found at extreme points of the unit simplex $S(1,m)$ of all distributions $(q_1, ..., q_m)$, we get

$$\begin{aligned}
\underline{\mathbb{E}}^{(\varepsilon)}\mathbf{u}_r &= (1-\varepsilon)\sum_{i=1}^{m} u_{ri}\pi_i + \varepsilon \min_{q_i} \sum_{i=1}^{m} u_{ri} q_i \\
&= (1-\varepsilon)\sum_{i=1}^{m} u_{ri}\pi_i + \varepsilon \min_{i=1,\dots,m} u_{ri},
\end{aligned}$$

$$\begin{aligned}
\overline{\mathbb{E}}^{(\varepsilon)}\mathbf{u}_r &= (1-\varepsilon)\sum_{i=1}^{m} u_{ri}\pi_i + \varepsilon \max_{q_i} \sum_{i=1}^{m} u_{ri} q_i \\
&= (1-\varepsilon)\sum_{i=1}^{m} u_{ri}\pi_i + \varepsilon \max_{i=1,\dots,m} u_{ri}.
\end{aligned}$$

The intervals $[\underline{\mathbb{E}}^{(\varepsilon)}\mathbf{u}_r, \overline{\mathbb{E}}^{(\varepsilon)}\mathbf{u}_r]$ by different values of ε are nested. This implies that we can construct a fuzzy expected utility $\widetilde{\mathbb{E}}\mathbf{u}_r$ with the membership function $\mu(\varepsilon)$ such that $\mu(1) = 0$ and $\mu(0) = 1$, for instance, $\mu(\varepsilon) = 1 - \varepsilon$.

As pointed out by Seidenfeld and Wasserman in the discussion part of Walley's paper (Walley (1996a)), the imprecise Dirichlet model has the same lower and upper probabilities as the ε-contaminated model. If we denote $\pi_i = n_i/N$ and $\varepsilon = s/(N + s)$, then we obtain the same lower and upper bounds for probabilities of states of nature. The same also can be said about lower and upper expected utilities. Thus, we can look at the fuzzy decision problem from two points of view now and can differently interpret it.

7 Special cases of membership functions

7.1 The simplest membership function

Let us take the function $\mu(s) = (1 + s)^{-1}$. Then $s = \mu^{-1} - 1$. After substituting this function into (5) and integrating, we obtain

$$\begin{aligned} F_r &= \left(1 - \frac{1 - N + N \ln N}{(N-1)^2}\right) A_r + G_r \frac{1 - N + N \ln N}{(N-1)^2} \\ &= \lambda \sum_{i=1}^{m} u_{ri} n_i / N + (1 - \lambda) \left(\eta \min_{i=1,\dots,m} u_{ri} + (1 - \eta) \max_{i=1,\dots,m} u_{ri}\right), \end{aligned}$$

where

$$\lambda = 1 - \frac{1 - N + N \ln N}{(N-1)^2}.$$

In sum, we have a very simple expression for F_r strongly depending on the total number of observations N.

Let us consider a case of complete ignorance about states of nature. Before making any observations, it can be stated $n_i = N = 0$. Then there holds

$$\lambda = 1 - \lim_{N \to 0} \frac{1 - N + N \ln N}{(N-1)^2} = 0.$$

This implies that $F_r = G_r$. In other words, the comparison index F_r corresponds to the Hurwicz criterion with optimism parameter $1 - \eta$, which is exploited when we do not have information about states of nature.

Let us consider a case when $N \to \infty$, i.e., there is a very large sample. The parameter λ is 1 in this case. Since n_i/N by $N \to \infty$ is the precise probability π_i of the i-th state, then we can write

$$F_r = \sum_{i=1}^{m} u_{ri} \pi_i.$$

One can see from the above that the proposed method for decision making strongly depends on statistical data about states of nature. In "extreme" cases when we do not observe states of nature $N = 0$ or we have a very large number of observations $N \to \infty$, the method is reduced to the well-known criteria which correspond to these cases. In fact, we get a frequency-based type of Hodges-Lehmann criterion (Hodges and Lehmann (1952)), which has been proposed in classical decision theory as a compromise between the Bayesian and the minimax approach.

7.2 The hinge membership function

Let us take the hinge linear function $\mu(s) = \max(0, 1 - s/a)$ which is often used in statistical machine learning methods (Vapnik (1998)). Then $s = a(1 - \mu)$. The parameter a can be interpreted as some limited value of s because $s = a$ when $\mu = 0$. After substituting this function into (5) and integrating with integration limits 0 and a, we obtain

$$\begin{aligned} F_r &= A_r + D_r \left(1 - \frac{N}{a} \ln \frac{N+a}{N}\right) \\ &= \lambda \sum_{i=1}^{m} u_{ri} n_i / N + (1 - \lambda) \left(\eta \min_{i=1,\dots,m} u_{ri} + (1 - \eta) \max_{i=1,\dots,m} u_{ri}\right), \end{aligned}$$

where

$$\lambda = \frac{N}{a} \ln \frac{N+a}{N}.$$

If $N = 0$, then $\lambda \to 0$. If $N \to \infty$, then $\lambda \to 1$ for arbitrary a. In sum, we have the same frequency-based type of Hodges-Lehmann criterion, but with the different parameter λ depending on a.

7.3 Membership functions in the framework of robust models

Let us use now the ε-contaminated model and suppose that $\mu(\varepsilon) = 1 - \varepsilon$. Since $\varepsilon = s/(N + s)$, then $\mu(s) = N/(N + s)$. Hence $s = N(\mu^{-1} - 1)$. By substituting this function into (5) and integrating, we obtain

$$\begin{aligned} F_r &= A_r + N D_r \int_0^\infty N(N+s)^{-3} \mathrm{d}s = A_r + D_r/2 \\ &= \frac{1}{N} \sum_{i=1}^{m} u_{ri} n_i - \frac{1}{2N} \sum_{i=1}^{m} u_{ri} n_i + \frac{1}{2} \left(\eta \min_{i=1,\dots,m} u_{ri} + (1 - \eta) \max_{i=1,\dots,m} u_{ri}\right) \\ &= \frac{1}{2} \sum_{i=1}^{m} u_{ri} n_i / N + \frac{1}{2} \left(\eta \min_{i=1,\dots,m} u_{ri} + (1 - \eta) \max_{i=1,\dots,m} u_{ri}\right). \end{aligned}$$

This special case shows that the "compromise" between Hurwicz criterion and expected utility does not depend on N. Therefore, the imprecise Dirichlet model is more preferable in comparison with the ε-contaminated model in decision making

Table 1 Values of the utility function u_{rj}

the number of repairmen	ϑ_1	ϑ_2	ϑ_3	ϑ_4
1 (a_1)	30	19	14	10
2 (a_2)	21	18	16	21
3 (a_3)	5	14	20	25

because it is difficult to determine how to change the value of ε when statistical data accumulate.

It is interesting to note that the above special case can be easily extended. Let us take the function $\mu(s) = N^t/(N+s)^t$, t is some positive integer. This function satisfies the introduced conditions for membership functions. By substituting this function into (5) and integrating, we obtain

$$F_r = A_r + ND_r \int_0^\infty \frac{N^t}{(N+s)^{t+2}} \mathrm{d}s = A_r + D_r \frac{1}{(t+1)}$$
$$= \frac{t}{(t+1)} \sum_{i=1}^{m} u_{ri} n_i / N + \frac{1}{(t+1)} \left(\eta \min_{i=1,\dots,m} u_{ri} + (1-\eta) \max_{i=1,\dots,m} u_{ri} \right).$$

We again get a linear combination of Hurwicz criterion and the expected utility with parameter $t/(t+1)$.

8 Numerical example

To illustrate the results of this paper, we solve the following numerical example. Let us consider a problem related to a number of repairmen for repairing equipment. There are available 40 units of equipment which can fail. Four states of nature are defined by the number of failed units during a certain time period: (ϑ_1) 0,...,10 units fail; (ϑ_2) 11,...,20 units fail; (ϑ_3) 21,...,30 units fail; (ϑ_4) 31,...,40 units fail. The maximal number of repairmen is 3.

The utility function is given in Table 1. Let us study different cases of N. At that, numbers n_i, $i = 1, ..., 4$, are generated by the random-number generator in accordance with the precise distribution π: $\pi_1 = 0.22$, $\pi_2 = 0.43$, $\pi_3 = 0.29$, $\pi_4 = 0.06$. We also use this distribution to consider the case $N \to \infty$.

This is a typical decision problem which is solved under the assumption of known probabilities of failures. However, it is difficult to get this information in many cases. Of course, approximate values of the probabilities can be used at times, but this way for solving the problem can lead to incorrect or non-optimal solutions. Therefore, we apply the proposed method and consider the fuzzy decision problem.

Membership functions of fuzzy sets $\widetilde{\mathbb{E}}\mathbf{u}_1$ (curve 1), $\widetilde{\mathbb{E}}\mathbf{u}_2$ (curve 2), and $\widetilde{\mathbb{E}}\mathbf{u}_3$ (curve 3) are shown in Fig. 1 ($N = 3$), Fig. 2 ($N = 5$), Fig. 3 ($N = 10$), Fig. 4 ($N = 30$) under condition $\mu(s) = (1+s)^{-1}$. It can be seen from the pictures that the fuzziness of $\widetilde{\mathbb{E}}\mathbf{u}_1$, $\widetilde{\mathbb{E}}\mathbf{u}_2$, $\widetilde{\mathbb{E}}\mathbf{u}_3$ decreases as the number N increases.

Table 2 shows how the ranking indices F_r depend on the caution parameter η. It is obvious that F_r does not depend on η and the optimal action is a_1 in the case of complete information ($N \to \infty$) . Optimal actions by different values of η and N are shown in Table 3.

Table 2 Fuzzy sets $\widetilde{\mathbb{E}}\mathbf{u_1}$, $\widetilde{\mathbb{E}}\mathbf{u_2}$, $\widetilde{\mathbb{E}}\mathbf{u_3}$ and ranking indices F_1, F_2, F_3 as functions of η by various N

N	(n_1, n_2, n_3, n_4)	$\widetilde{\mathbb{E}}\mathbf{u}_i$	F_1	F_2	F_3
0	$(0,0,0,0)$		$30 - 20\eta$	$21 - 5\eta$	$25 - 20\eta$
3	$(0,2,1,0)$	Fig.1	$21.4 - 6.5\eta$	$18.5 - 1.6\eta$	$18.9 - 6.5\eta$
5	$(2,2,1,0)$	Fig.2	$24.3 - 5.1\eta$	$19.3 - 1.3\eta$	$15.0 - 5.1\eta$
10	$(2,5,2,1)$	Fig.3	$21.2 - 3.5\eta$	$18.9 - 0.9\eta$	$16.3 - 3.5\eta$
30	$(6,14,9,1)$	Fig.4	$20.3 - 1.7\eta$	$18.3 - 0.4\eta$	$15.3 - 1.7\eta$
∞			19.43	18.26	14.42

Table 3 Optimal actions by different values of η and N

N	η				
	0	0.25	0.5	0.75	1
0	a_1	a_1	a_1	a_2	a_3
3	a_1	a_1	a_1	a_2	a_2
5	a_1	a_1	a_1	a_1	a_1
10	a_1	a_1	a_1	a_1	a_2
30	a_1	a_1	a_1	a_1	a_1
∞	a_1	a_1	a_1	a_1	a_1

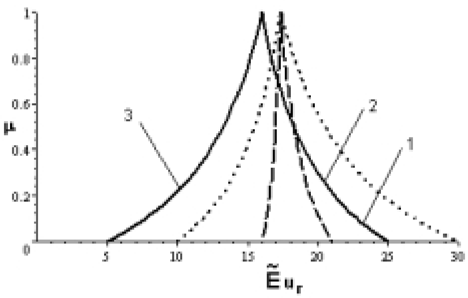

Figure 1 Fuzzy sets $\widetilde{\mathbb{E}}\mathbf{u}_1$ (curve 1), $\widetilde{\mathbb{E}}\mathbf{u}_2$ (curve 2), and $\widetilde{\mathbb{E}}\mathbf{u}_3$ (curve 3) by $N = 3$

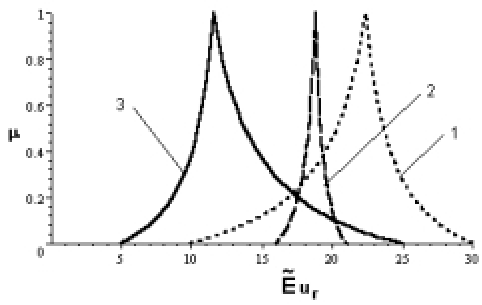

Figure 2 Fuzzy sets $\widetilde{\mathbb{E}}\mathbf{u}_1$ (curve 1), $\widetilde{\mathbb{E}}\mathbf{u}_2$ (curve 2), and $\widetilde{\mathbb{E}}\mathbf{u}_3$ (curve 3) by $N = 5$

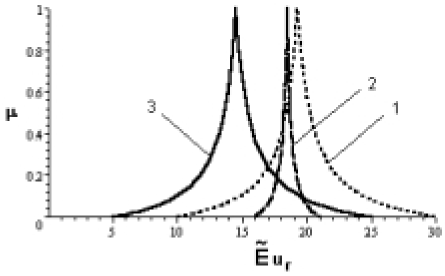

Figure 3 Fuzzy sets $\widetilde{\mathbb{E}}\mathbf{u}_1$ (curve 1), $\widetilde{\mathbb{E}}\mathbf{u}_2$ (curve 2), and $\widetilde{\mathbb{E}}\mathbf{u}_3$ (curve 3) by $N = 10$

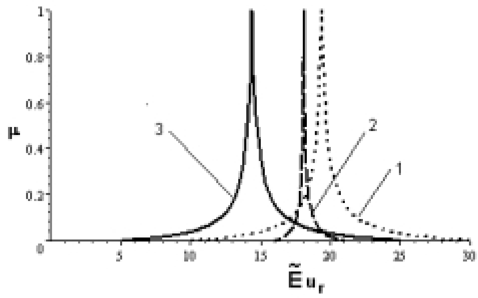

Figure 4 Fuzzy sets $\widetilde{\mathbb{E}}\mathbf{u}_1$ (curve 1), $\widetilde{\mathbb{E}}\mathbf{u}_2$ (curve 2), and $\widetilde{\mathbb{E}}\mathbf{u}_3$ (curve 3) by $N = 30$

The following inferences can be made based on the numerical results.

1. When the number of observations of states of nature is small, we have a large fuzziness of fuzzy expected utilities (see Fig. 1 and Fig. 2). This shows that the fuzziness reflects the amount of statistical data.

2. The ranking indices F_r weakly depend on the caution parameter η by large values of N.

3. There is a value of N such that the action does not depend on the caution parameter η because there holds $\lambda \to 1$ (see the last rows of Table 3 by $N = 30$ and $N \to \infty$).

9 Conclusion

A simple decision problem taken into account numbers of observations of states of nature has been studied in the paper. It exploits fuzzy sets in order to avoid the subjective choice of the hyperparameter s of Walley's imprecise Dirichlet model. It should be noted that only a special case of the fuzzy ranking has been studied in detailed. Therefore, the method can be extended by applying various fuzzy ranking indices which take into account the form of membership functions.

The main idea underlying the proposed method can also be applied to more complex decision problems, like multi-criteria decision making. Indead, the weights of criteria in voting multi-criteria decision problems or relative importance of criteria can be regarded as fuzzy numbers elicited from simplest expert judgments by means of the proposed method. The idea can also be extended on decision problems with the combined objective and subjective initial information about states of nature, and decision problems with imprecise utilities. However, these are topics for further work.

Another topic of further research is to extend the results obtained here to other optimality criteria which are used in imprecise decision problems (Troffaes (2007)), for example, Walley's maximality criterion (Walley (1991)). The problem can also be extended to a case of the randomized strategy when an optimal probability distribution defined on the set of actions is computed.

The proposed approach can also be applied to construction of fuzzy sets on the basis of statistical observations. It should be noted that the idea to construct fuzzy sets by using statistical observations in order to model the imprecision and incompleteness of statistical data can be extended on the case of the continuous set of states of nature. By having only some observations of the corresponding random variables, we can construct fuzzy sets by using, for instance, the well-known confidence Kolmogorov-Smirnov or Anderson-Darling bands (Frey (2009)) with certain confidence probabilities. This is an interesting direction for further research. Moreover, it is important to consider different imprecise Bayesian models (Quaeghebeur and de Cooman (2005)) for constructing the fuzzy sets in future.

Acknowledgement

The authors would like to express their appreciation to the anonymous referees whose very valuable comments have improved the paper.

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