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Imprecise Reliability of General Structures

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Abstract. The paper discusses the important aspects of the reliability of systems with an imprecise general model of the structure function. It is assumed that the information about reliability behavior of components is restricted by the mean levels of component performance. In this case the classical reliability theory cannot provide a way for analyzing the reliability of systems. The theory of imprecise probabilities may be a basis in developing a general reliability theory which allows us to solve such problems. The basic tool for computing new reliability measures is the natural extension which can be regarded as a linear optimization problem. However, the linear programming computations will become impracticable when the number of components in the system is large. Therefore, the main aim of the paper is to obtain explicit expressions for computing the system reliability measures. We analyze the reliability of general structures and typical systems. The numerical examples illustrate the usefulness of the presented approach to reliability analyzing.

Keywords: reliability, imprecise probabilities, multistate systems, previsions, upper and lower probabilities.

1 Introduction

Classical reliability theory assumes that all probabilities are precise, that is, that every probability involved is perfectly determinable. If the information we have about the functioning of components and systems is based on a statistical analysis, then a probabilistic uncertainty model should be used in order to mathematically represent and manipulate that information. However, the probabilistic assumption may be unreasonable in a large number of cases. It very often happens that probabilities cannot be determined exactly, either due to measurement imperfections, or due to more fundamental reasons, such as insufficient available information. Moreover, in some cases, the information we have about the functioning of components and systems is not based on statistics, but is of a linguistic nature, i.e. the information is conveyed by statements in natural language. A part of the reliability assessments may be supplied by experts. For example, assessments of experts may have such the forms as “The mean level of component performance is more than 0.1”, “Failure before 10 hours is probable”. Other assessments may be made by the user of the system during the experimental service. It should be noted that in reliability theory there are a lot of analytical estimates which can be used as an additional information, for example, independence of component failures. Lastly, we may know nothing at all about the reliability of systems or components. Thus, the reliability assessments that are combined to describe systems and components may come from various sources. How to compute the system reliability measures on the basis of the imprecise information on the component reliability?

One of the ways to model partial or complete ignorance is through a concept called fuzzy reliability [1–6]. However, in spite of successful application of the possibility theory for the reliability analysis of various systems, possibility measures cannot model many types of uncertainty.

To develop a general reliability theory taking into account the various sources of information, de Cooman proposed using the theory of imprecise probabilities (also called the theory of lower previsions [7] and the theory of interval statistical models [8]) introduced and developed by Walley [9]. A general framework for the theory of imprecise probabilities is provided by upper and lower previsions. They can model a very wide variety of kinds of uncertainty, partial information, and ignorance. According to this theory, possibility measures are a special type of upper probabilities [10, 7, 11]. Walley’s theory of imprecise probabilities is arguably the most satisfactory of all current theories of uncertain reasoning from a fundamental point of view [12]. The rules used in the theory of lower previsions, which are based on a general procedure called natural extension, can be applied to various measures, including possibility measures. Therefore, this theory might be a basis to develop a general reliability theory.

In this paper we try to apply the theory of imprecise probabilities to the reliability analysis of multistate and continuum systems. It is assumed that the information about reliability behavior of components is restricted by bounds of the mean levels of component performance. The basic tool for computing new reliability measures is the natural extension which can be regarded as a linear optimization problem. However, the linear programming computations will become impracticable when the number of components in the system is large. Therefore, the main aim of the paper is to obtain the explicit expressions for computing the system reliability measures. The paper is organized as follows. Section 2 is a brief introduction to the theory of imprecise probabilities. The basic definitions of systems with the general structure function and notions of the general reliability theory are considered in Sec. 3. It is shown in this section how to apply the natural extension to the system reliability calculus. The important properties of the natural extension are presented in Sec. 4. These properties allow us to simplify the optimization problems associated with the natural extension. In Sec. 5 we analyze the reliability of the general structures under condition that the component reliability is determined by the upper and lower mean levels of performance. In this section, series and parallel systems are studied.

2 Gambles and Previsions

Let us briefly review the basic concepts of the theory of imprecise probabilities. All the definitions and results introduced in this section can be found in [9, 7, 11].

Let Ω be a set called the *possibility space* or the *universe of discourse*. It can be interpreted as the set of the mutually exclusive possible outcomes of a specific experiment. A real-valued mapping X on Ω will be called a *gamble* on Ω iff it is bounded, i.e. if $\sup X = \sup\{X(\omega) : \omega \in \Omega\}$ and $\inf X = \inf\{X(\omega) : \omega \in \Omega\}$ are finite real numbers. It is interpreted as a reward which will be paid, after observing the value of X. For example, X might denote the amount of failures that will occur during a predefined period of time. The set of the gambles on Ω is denoted by $\mathcal{L}(\Omega)$. The *upper prevision* of a gamble X, denoted by $\overline{M}(X)$, is a real-valued function on a class of gambles $\mathcal{K} \subseteq \mathcal{L}(\Omega)$. It can be interpreted as an infimum selling price for X. The *conjugate lower prevision* $\underline{M}(X)$ is defined on $-\mathcal{K} = \{-X : X \in \mathcal{K}\}$ by $\underline{M}(X) = -\overline{M}(-X)$, $X \in -\mathcal{K}$. It can be interpreted as a supremum buying price for X. It is often natural to regard $\underline{M}(X)$ and $\overline{M}(X)$ as lower and upper bounds for some ideal price $M(X)$ that is not known precisely. Note that if we have an interval $[\underline{M}(X), \overline{M}(X)]$ for the gamble $X \in \mathcal{K}$, then we can specify an interval $[-\overline{M}(X), -\underline{M}(X)]$ for the gamble $X \in -\mathcal{K}$. This implies that $\overline{M}(X) = -\underline{M}(-X)$ and $\underline{M}(X) = -\overline{M}(-X)$.

There is a special class of gambles that assume only values in $\{0, 1\}$. A subset $A = \{\omega \in \Omega : X(\omega) = 1\}$ of Ω will be called an event and the *upper* and *lower probabilities* of the event A are defined to be the upper and lower previsions of $0-1$-valued gambles. They are related by $\overline{P}(A) = 1 - \underline{P}(A^c)$, where A^c denotes the complement of A. Upper and lower previsions can also be regarded as generalizations of upper and lower probabilities.

To model complete ignorance the *vacuous* previsions $\underline{M}(X) = \inf X$ and $\overline{M}(X) = \sup X$ are used.

Suppose that there exist the upper $\overline{F}_X$ and lower $\underline{F}_X$ distribution functions of X. Then the upper and lower previsions can be interpreted as upper and lower expectations

$$\overline{M}(X) = \int_{-\infty}^{\infty} x \mathrm{d}\overline{F}_X(x), \ \underline{M}(X) = \int_{-\infty}^{\infty} x \mathrm{d}\underline{F}_X(x).$$

In the theory of imprecise probabilities, there are three fundamental principles: avoiding sure loss, coherence and natural extension. Avoiding sure loss can be considered as a rationality condition. The lower prevision $\underline{M}(X)$ avoids sure loss if $\sup X \geq \underline{M}(X)$, i.e. You should not be willing to pay more for X than the supremum amount You can get back. Similarly, the upper prevision $\overline{M}(X)$ avoids sure loss if $\overline{M}(X) \geq \inf X$. Avoiding sure loss implies that $\underline{M}(X) \leq \overline{M}(X)$ and $\underline{P}(A) + \underline{P}(A^c) \leq 1$. Coherence characterizes a kind of self-consistency of previsions. Like avoiding sure loss, coherence is proposed as a criterion of rationality. However, it is a much stronger condition than avoiding sure loss [9]. The basic rules can be derived from the coherence principle, for example:

1. $\inf\{X\} \leq \underline{M}(X) \leq \overline{M}(X) \leq \sup\{X\}$
2. $\underline{M}(cX) = c\underline{M}(X)$, $c > 0$
3. $\underline{M}(X)+\underline{M}(Y) \leq \underline{M}(X+Y) \leq \underline{M}(X)+\overline{M}(Y) \leq \overline{M}(X+Y) \leq \overline{M}(X)+\overline{M}(Y)$.

Natural extension is a general mathematical procedure for calculating new previsions from the initial judgements. It produces a coherent overall model from an arbitrary collection of imprecise probability judgements and may be seen as the basic constructive step in statistical reasoning. Natural extension summarizes the buying prices for gambles that are implied by $\underline{M}$ and $\overline{M}$ through the linear operations involved in the definition of coherence. For computing new previsions $\overline{M}(X_s)$ and $\underline{M}(X_s)$ from an available set of the previsions $\overline{M}(X_i)$ and $\underline{M}(X_i)$, $i = 1, \ldots, n$, the natural extension can be used in the following form:

$$\overline{M}(X_s) = \inf_{c_0, c_i, d_i} \left(c_0 + \sum_{i=1}^{n} \left(c_i \overline{M}(X_i) - d_i \underline{M}(X_i) \right) \right), \tag{1}$$

where c_i, $d_i \in \mathbf{R}^+$, $c_0 \in \mathbf{R}$, $i = 1, \ldots, n$, and

$$c_0 + \sum_{i=1}^{n} (c_i X_i - d_i X_i) \geq X_s,\ \inf X_i \leq X_i \leq \sup X_i,\ i = 1, \ldots, n.$$

$$\underline{M}(X_s) = -\overline{M}(-X_s),$$

Equivalent definitions of avoiding sure loss, coherence and natural extension can be found in [8].

3 Structure Function is a Gamble, Mean Level of Performance is a Prevision

Let L be the set representing levels of component performance ranging from perfect functioning $\sup L$ to complete failure $\inf L$. A *general* model of the structure function of a system consisting of n multistate components was considered in [13]. It can be written as $S : L^n \to L$. If $L = \{0, 1\}$, we have a classical *binary* system; if $L = \{0, 1, \ldots, m\}$, we have a *multistate* system; if $L = [0, T]$, $T \in \mathbf{R}^+$, we have a *continuum* system. At arbitrary time t the i-th component may be in a state $x_i(t)$. This implies that the component is described by the random process $\{x_i(t), t \geq 0\}$, $x_i(t) \in L$. Then the probability distribution function of the ith component states at time t is defined as the mapping $F_i : L \to [0, 1]$ such that $F_i(r) = \Pr\{x_i(t) \geq r\}$ $\forall r \in L$. The state of the system is determined by states of its n components, i.e.

$$S = S(\mathbf{X}) = S(x_1, \ldots, x_n) \in L.$$

Then the probability distribution function of the system states at time t is defined as the mapping $F : L \to [0, 1]$ such that $F(r) = \Pr\{S(\mathbf{X}) \geq r\}$ $\forall r \in L$.

A system of n components with the structure function S is called a *monotone* multistate system if $S(\mathbf{X})$ is increasing in each argument and $S(r, \ldots, r) = r$ $\forall r \in L$. An arbitrary structure function of a monotone multistate system can be expressed through operations “min” and “max”. Properties of multistate systems can be found in [14].

Define the *mean level of component performance* $h_i(t)$ as $h_i(t) = E\{x_i(t)\}$. Here E is an expectation operator. For a system, we write the mean level of system performance $h(t)$ as $h(t) = E\{S(\mathbf{X})\}$.

Example 1. Consider a multistate system consisting of two components with the structure function $\{0,1,2\}^2 \to \{0,1,2\}$ given in Table 1. From Table 1 we can see that there holds $S(\mathbf{X}) = \min(x_1, x_2)$. Here

$$h_i(t) = \sum_{k=1}^{2} k \Pr\{x_i(t) = k\} = \sum_{k=1}^{2} \Pr\{x_i(t) \geq k\} = \sum_{k=1}^{2} F_i(k),$$

$$h(t) = \sum_{k=1}^{2} k \Pr\{S(x_1, x_2) = k\} = \sum_{k=1}^{2} \Pr\{S(x_1, x_2) \geq k\} = \sum_{k=1}^{2} F(k).$$

Table 1. Structure function of the series system

x_2 \ x_1	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

Example 2. Consider a system consisting of two components with the structure function $S : [0,1]^2 \to [0,1]$ such that $S(\mathbf{X}) = \min(x_1, x_2)$. Then the following hold:

$$h_i(t) = \int_0^1 F_i(z)\mathrm{d}z,$$

$$h(t) = \int_0^1 F(z)\mathrm{d}z = \int_0^1 F_1(z)F_2(z)\mathrm{d}z.$$

Suppose that the state probabilities of components are unknown. However, we can obtain lower $\underline{h}_i(t)$ and (or) upper $\overline{h}_i(t)$ bounds of mean levels of component performance from experts (in particular, $\underline{h}_i(t) = \overline{h}_i(t)$). How to find bounds $\underline{h}(t)$ and $\overline{h}(t)$ of mean levels of system performance? We can see that variables $x_i(t)$, $i = 1, \dots, n$, and $S(\mathbf{X})$ can be considered as gambles because their values are uncertain.

Suppose that there exist the upper $\overline{F}_i$ and lower $\underline{F}_i$ probability distribution functions of the ith component states at time t. Then there hold $\overline{h}_i(t) = E^*\{x_i(t)\}$ and $\underline{h}_i(t) = E_*\{x_i(t)\}$, where E^* and E_* are the expectation operators corresponding to upper and lower distribution functions, respectively. This implies that the lower (upper) mean levels of performance are the lower (upper) previsions and we can use the principles of the imprecise probabilities theory for computing $\underline{h}(t)$ and $\overline{h}(t)$. We assume that previsions $\underline{h}_i(t)$ and $\overline{h}_i(t)$, $i = 1, \dots, n$, satisfy the rationality conditions, i.e. they avoid sure loss and are coherent.

To model complete ignorance for the ith component the vacuous previsions $\underline{h}_i(t) = \inf L$ and $\overline{h}_i(t) = \sup L$ can be used. In a special case $L = \{0, 1\}$, we have the binary coherent system with the structure function $S : \{0, 1\}^n \to \{0, 1\}$. For this system, gambles x_i assume only values in $\{0, 1\}$, where 1 corresponds to the operating state. Therefore, the values $\underline{h}_i(t)$ and $\overline{h}_i(t)$ can be considered as lower and upper probabilities of the operating state of the ith component, $i = 1, \dots, n$. For example, if the statement "functioning of component is probable at time 10 h" conveys information about the reliability of the ith component, then $\underline{h}_i(10) = 0.5$ and $\overline{h}_i(10) = 1$ (the vacuous prevision). The values $\underline{h}(t)$ and $\overline{h}(t)$ are the lower and upper probabilities of the system operating state.

In the following, we shall omit variable t for brevity.

In order to use the available information for computing mean levels of system performance we can make various assumptions. For example, we may assume that the time to failure is a random variable governed by the exponential probability distribution and apply methods of the classical reliability theory. We may also assume that the time to failure is a fuzzy variable and apply known methods of the fuzzy reliability theory [1–3]. However, all the additional assumptions may be in contradiction with the real behavior of systems or may be unreasonable in a wide scope of cases. Therefore, for computing the system reliability measures, we have to take into account only the available information. This can be done by means of rules of the imprecise probabilities theory.

For computing the lower prevision $\underline{h}$ from an available set of the component previsions $\underline{h}_i(t)$ and $\overline{h}_i(t)$, $i = 1, \dots, n$, the natural extension can be used in the following form:

$$\underline{h} = \sup_{c_0, c_i, d_i} \left(c_0 + \sum_{i=1}^{n} \left(c_i \underline{h}_i - d_i \overline{h}_i \right) \right), \tag{2}$$

where c_i, $d_i \in \mathbf{R}^+$, $c_0 \in \mathbf{R}$, $i = 1, \dots, n$, and

$$c_0 + \sum_{i=1}^{n} (c_i x_i - d_i x_i) \leq S(\mathbf{X}), \ x_i \in L.$$

For computing the upper prevision $\overline{h}$, the natural extension can be used in the following form:

$$\overline{h} = \inf_{c_0, c_i, d_i} \left(c_0 + \sum_{i=1}^{n} \left(c_i \overline{h}_i - d_i \underline{h}_i \right) \right), \tag{3}$$

$$c_0 + \sum_{i=1}^{n} (c_i x_i - d_i x_i) \geq S(\mathbf{X}), \ x_i \in L.$$

Here we do not assume that events of failures of components are independent, i.e. mean levels of one component performance would not change if we learned whether or not the failure of other component occurred. Thus, we do not have to know whether the observations are independent.

So, new previsions $\overline{h}$ and $\underline{h}$ can be computed as solutions to linear programming problems. However, the linear programming computations will become impracticable when the number of components in the system is large. Therefore, we will consider several special cases, for which natural extensions can be computed explicitly, without linear programming.

4 Preliminary Results

Denote $\mathbf{C} = (c_1, \dots, c_n)$, $\mathbf{X} = (x_1, \dots, x_n)^{\mathrm{T}}$,

$$D = \prod_{i=1}^{n} [0, T] \subset \mathbf{R}^n,$$

$$D^* = \{(T^{(i_1)}, \dots, T^{(i_n)}) \mid i_j = 0, 1,\ j = 1, \dots, n,\ T^{(0)} = 0,\ T^{(1)} = T\}.$$

We assume that the inequality $\mathbf{X} < \mathbf{Y}$ means that $\forall i$, $x_i < y_i$, x_i and y_i are i-th components of vectors $\mathbf{X}$ and $\mathbf{Y}$, respectively.

Theorem 1. *Suppose that*
1. $S(\mathbf{X}) \geq 0$, $\mathbf{X} \in D$*;*
2. $S(\mathbf{X})$ *is a non-decreasing continuous function;*
3. For each $\mathbf{X}$*, there exists a number* i_0 *such that* $S(\mathbf{X}) = x_{i_0}$*, where* x_{i_0} *is a component of the vector* $\mathbf{X}$*.*
Then
1. the system of inequalities $c_0 + \mathbf{C}\mathbf{X} \geq S(\mathbf{X})$, $\mathbf{C} \geq 0$*, is valid for* $\forall \mathbf{X} \in D$ *iff it is valid for* $\forall \mathbf{X} \in D^*$*;*
2. the system of inequalities $c_0 + \mathbf{C}\mathbf{X} \leq S(\mathbf{X})$, $\mathbf{C} \geq 0$*, is valid for* $\forall \mathbf{X} \in D$ *iff it is valid for* $\forall \mathbf{X} \in D^*$*.*

Proof. (*If part*). The proof is obvious because $D^* \subset D$.
(*Only if part*). Denote

$$D(i_1, \dots, i_n) = \{\mathbf{X} : x_{i_1} \geq x_{i_2} \geq \dots \geq x_{i_n}\} \subset \mathbf{R}^n.$$

Without loss of generality, suppose $\mathbf{X} \in D(1, 2, \dots, n)$. Introduce for $D(1, \dots, n)$ a basis with non-negative coefficients d_i, $i = 1, \dots, n$. It can be the following set of vectors $\{\mathbf{a}_1, \dots, \mathbf{a}_n\} \in D(1, 2, \dots, n)$:

$$\mathbf{a}_i = (\underbrace{T, \dots, T}_{i \text{ times}}, \underbrace{0, \dots, 0}_{n-i \text{ times}})^{\mathrm{T}},\ i = 1, \dots, n$$

Note that $\mathbf{a}_k \in D^*$. Then we have to prove that if the inequality $c_0 + \mathbf{C}\mathbf{a}_k \geq S(\mathbf{a}_k)$ is valid, then $c_0 + \mathbf{C}\mathbf{X} \geq S(\mathbf{X})$ is valid.

Note that $\mathbf{X} = \sum_{i=1}^{n} d_i \mathbf{a}_i$. Indeed, there holds

$$x_i = \sum_{k=i}^{n} d_k T. \tag{4}$$

Hence

$$d_n = \frac{x_n}{T},\ d_{n-1} = \frac{x_{n-1} - x_n}{T}, \dots,\ d_1 = \frac{x_1 - x_2}{T}.$$

Since $x_{k-1} - x_k \geq 0$ for all $k = 2, \dots, n$, ($\mathbf{X} \in D(1, 2, \dots, n)$), then $d_k \geq 0$ for all $k = 1, \dots, n$. Moreover, there holds $\sum_{i=1}^{n} d_i = x_1/T \leq 1$.

Let $c_0 + \mathbf{C}\mathbf{a}_k \geq S(\mathbf{a}_k)$ be valid. By substituting $\mathbf{X} = \sum_{i=1}^{n} d_i \mathbf{a}_i$ into $c_0 + \mathbf{C}\mathbf{X} \geq S(\mathbf{X})$, we obtain

$$c_0 + \mathbf{C}\mathbf{X} = c_0 + \sum_{k=1}^{n} d_k \mathbf{C}\mathbf{a}_k \geq c_0 + \sum_{k=1}^{n} d_k (S(\mathbf{a}_k) - c_0)$$

$$= c_0 + \sum_{k=1}^{n} d_k S(\mathbf{a}_k) - c_0 \sum_{k=1}^{n} d_k = c_0 \left(1 - \sum_{k=1}^{n} d_k\right) + \sum_{k=1}^{n} d_k S(\mathbf{a}_k).$$

Let us prove that $S(\mathbf{X}) = \sum_{k=1}^{n} d_k S(\mathbf{a}_k)$. Suppose that $S(\mathbf{X}) = x_j$, $\mathbf{X} \in D(1, 2, \dots, n)$. We have to prove that for $\forall \mathbf{X} \in D(1, 2, \dots, n)$ there holds $S(\mathbf{X}) = x_j$. We shall divide the set $D(1, 2, \dots, n)$ into n disjoint subsets

$$D_i = \{\mathbf{X} \in D(1, 2, \dots, n) : S(\mathbf{X}) = x_i\},\ i = 1, \dots, n.$$

Since the function $S(\mathbf{X})$ is continuous, then subsets D_i are closed and there holds $D(1, 2, \dots, n) = \bigcup_{i=1}^{n} D_i$. Now we have to prove that one of the subsets D_i, $i = 1, \dots, n$, is non-empty and other subsets are empty. In this case there holds $D(1, 2, \dots, n) = D_i$. Denote $\bigcup_{i=2}^{n} D_i = D_1^*$. Then $D(1, 2, \dots, n) = D_1 \cup D_1^*$. Note that D_1^* is closed as a union of the finite number of closed sets. Moreover, $D_1 \cap D_1^* = \emptyset$. We have to prove that $D_1 = \emptyset$, $D_1^* \neq \emptyset$ or $D_1^* = \emptyset$, $D_1 \neq \emptyset$. Conversely, assume that $D_1 \neq \emptyset$ and $D_1^* \neq \emptyset$. Then there exist $\mathbf{X}_1 \in D_1$ and $\mathbf{X}_2 \in D_1^*$. A line segment joining points $\mathbf{X}_1$ and $\mathbf{X}_2$ contains a point $\mathbf{X} = (1 - t)\mathbf{X}_1 + t\mathbf{X}_2$, $t \in [0, 1]$. This line segment belongs to $D(1, 2, \dots, n)$ due to convexity of $D(1, 2, \dots, n)$. There exists $t_0 \in (0, 1)$ such that if $t \leq t_0$, then $\mathbf{X} \in D_1$, if $t > t_0$, then $\mathbf{X} \in D_1^*$. Indeed, by $t \leq t_0$ we consider a sequence $t_n \to t_0 + 0$. Then for all the points $\mathbf{X}_n$ there holds $\mathbf{X}_n \in D_1^*$ and a limit point belongs to D_1. This implies that D_1^* is not closed, a contradiction. A similar conclusion can be obtained for the case $t > t_0$. If $D_1^* = \emptyset$, then $D(1, 2, \dots, n) = D_1$ and $S(\mathbf{X}) = x_1$. If $D_1 = \emptyset$, then $D(1, 2, \dots, n) = D_1^*$. By dividing the set D_1^* and repeating this argument, we obtain one set.

Since $\mathbf{a}_i \in D(1, 2, \dots, n)$, then $S(\mathbf{a}_1) = \cdots = S(\mathbf{a}_{j-1}) = 0$, $S(\mathbf{a}_j) = \cdots = S(\mathbf{a}_n) = T$. It follows from (4) that

$$\sum_{k=1}^{n} d_k S(\mathbf{a}_k) = \sum_{k=j}^{n} d_k T = x_j = S(\mathbf{X}).$$

Since $c_0 \geq 0$ and $\sum_{k=1}^{n} d_k \leq 1$, then there hold

$$c_0 + \mathbf{C}\mathbf{X} \geq c_0 \left(1 - \sum_{k=1}^{n} d_k\right) + S(\mathbf{X}) \geq S(\mathbf{X}).$$

This completes the proof.

The second conclusion is proved similarly by taking into account the condition $c_0 \leq 0$.

Theorem 1 is very important because it allows us to regard the systems with the general structure function as a two-state system. The conclusion of Theorem 1 essentially reduces the complexity of optimization problems (2) and (3).

Theorem 2. *Suppose that $S(\mathbf{X})$ is a non-decreasing function and $S(\mathbf{X}) \geq 0$, $\mathbf{X} \in D$. Then problem (2) is equivalent to the following problem*

$$\underline{h} = \sup_{\mathbf{C} \geq 0, c_0} \left(c_0 + \sum_{i=1}^{n} c_i \underline{h}_i \right), \tag{5}$$

subject to $c_0 + \mathbf{CX} \leq S(\mathbf{X}),\ \mathbf{X} \in D$.

Proof. Denote $e_i = c_i - d_i$, $e_i \in \mathbf{R}$. If $x_i \geq 0$ and $S(\mathbf{X}) \geq 0$, then problem (2) can be rewritten as follows:

$$\underline{h} = \sup_{c_0, \mathbf{C} \geq 0, e_i} \left(c_0 + \sum_{i=1}^{n} \left(c_i \underline{h}_i - c_i \overline{h}_i + e_i \overline{h}_i \right) \right),$$

$$c_0 + \sum_{i=1}^{n} e_i x_i \leq S(\mathbf{X}),\ x_i \in L.$$

Since $d_i \geq 0$, then $c_i \geq e_i$. Since $\overline{h}_i - \underline{h}_i \geq 0$, then the objective function is maximal when $c_i = \min\{0, e_i\}$, $i = 1, \ldots, n$. Hence

$$\underline{h} = \sup_{c_0, e_i} \left(c_0 + \sum_{i=1}^{n} \min(e_i \overline{h}_i, e_i \underline{h}_i) \right), \tag{6}$$

$$c_0 + \sum_{i=1}^{n} e_i x_i \leq S(\mathbf{X}),\ x_i \in L,\ e_i \in \mathbf{R}.$$

Assume that $c_i = e_i$, $i = 1, \ldots, n$. Note that $(c_0, 0, \ldots, 0)$ is a feasible point of problem (6) and for this point the value of the objective function is equal to c_0. Denote $N = \{1, 2, \ldots, n\}$ and

$$\widetilde{c}_k = \begin{cases} c_k,\ k \notin J \\ 0,\ \ k \in J \end{cases},\ \widetilde{x}_k = \begin{cases} x_k,\ k \notin J \\ 0,\ \ k \in J \end{cases}.$$

If $(c_0, c_1, \ldots, c_n)$ is a feasible solution and for values of a set $J \subset N$ there holds $c_k < 0$ for all $k \in J$, then $(c_0, \widetilde{c}_1, \ldots, \widetilde{c}_n)$ is a feasible solution corresponding to a greater value of the objective function. Indeed, assuming an inequality $c_0 + \mathbf{CX} \leq S(\mathbf{X})$ that $x_k = 0$ for all $k \in J$ and by using the properties of the function S, we obtain

$$c_0 + \sum_{i \in N} \widetilde{c}_i x_i = c_0 + \sum_{i \in N \setminus J} c_i x_i \leq S(\widetilde{x}_1, \ldots, \widetilde{x}_n) \leq S(x_1, \ldots, x_n).$$

Thus, we can accept that $c_k \geq 0$ for all $k \in J$. Since the set J can be arbitrary, then there holds $\mathbf{C} \geq 0$ and problem (6) is equivalent to problem (5), as was to be proved.

Theorem 2 implies that the lower prevision of a system depends only on lower previsions of primary gambles. In other words, the lower mean level of system performance depends only on lower mean levels of component performances.

Theorem 3. *Suppose that $S(\mathbf{X})$ is a non-decreasing function and $S(\mathbf{X}) \geq 0$, $\mathbf{X} \in D^*$. Then problem (3) is equivalent to the following problem*

$$\overline{h} = \inf_{\mathbf{C} \geq 0, c_0} \left(c_0 + \sum_{i=1}^{n} c_i \overline{h}_i \right), \tag{7}$$

subject to $c_0 + \mathbf{CX} \geq S(\mathbf{X})$, $\mathbf{X} \in D^*$.

Proof. Problem (3) can be rewritten as follows (see the proof of Theorem 2):

$$\overline{h} = \inf_{c_0, c_i} \left(c_0 + \sum_{i=1}^{n} \max(c_i \overline{h}_i, c_i \underline{h}_i) \right),$$

$$c_0 + \sum_{i=1}^{n} c_i x_i \geq S(\mathbf{X}),\ x_i \in L,\ c_i \in \mathbf{R}.$$

According to Theorem 1, the optimal solution can be found only for $\mathbf{X} \in D^*$. Therefore, we can rewrite problem (3) as follows:

$$\overline{h} = \inf_{c_0, c_i} \left(c_0 + \sum_{i=1}^{n} \max(c_i \overline{h}_i, c_i \underline{h}_i) \right), \tag{8}$$

$$c_0 + \sum_{i \in I} c_i T_i \geq S(\mathbf{T}_I),$$

where I is an arbitrary set of indices; $\mathbf{T}_I$ is a vector with non-zero components T_i, $i \in I$.

If I is empty, then $S(\mathbf{T}_I) = 0$ and $c_0 \geq 0$. Let us prove that the optimal solution is achieved by $c_i \geq 0$, $i = 1, \dots, n$. Let $C^* = (c_0^*, c_1^*, \dots, c_n^*)$ be an optimal solution. Denote $I_- = \{i : c_i^* \leq 0\}$, $I_+ = \{i : c_i^* \geq 0\}$. Then for any I, the following inequality is valid:

$$c_0^* + \sum_{i \in I} c_i^* T_i \geq S(\mathbf{T}_I). \tag{9}$$

Hence

$$\overline{h} = c_0^* + \sum_{i \in I_+} c_i^* \overline{h}_i + \sum_{i \in I_-} c_i^* \underline{h}_i.$$

Note that inequality (9) is valid for the empty set I. Consider a solution $\widetilde{C} = (\widetilde{c}_0, \widetilde{c}_1, \dots, \widetilde{c}_n)$, where

$$\widetilde{c}_0 = c_0^* + \sum_{i \in I_-} c_i^* \underline{h}_i,\ \widetilde{c}_i = \begin{cases} c_i^*,\ i \in I_+ \\ 0,\ \ i \in I_- \end{cases}.$$

Let us show that $\widetilde{C}$ is the feasible solution, i.e.

$$\widetilde{c}_0 + \sum_{i \in I_+} \widetilde{c}_i x_i + \sum_{i \in I_-} \widetilde{c}_i x_i \geq S(\mathbf{X}).$$

Let $J_0 = \{i : x_i = 0\}$, $J_+ = \{i : x_i = T_i\}$. Denote $\widehat{\mathbf{T}} = (x_1, \dots, x_n)$, where

$$x_i = \begin{cases} T_i, \, i \in J_+ \\ 0, \quad i \in J_0 \end{cases}.$$

Denote $\widetilde{\mathbf{T}} = (\widetilde{T}_1, \dots, \widetilde{T}_n)$, where

$$\widetilde{T}_i = \begin{cases} T_i, \, i \in J_+ \cup (I_- \cap J_0) \\ 0, \quad i \notin J_+ \cup (I_- \cap J_0) \end{cases}.$$

Then

$$c_0^* + \sum_{i \in I_-} c_i^* T_i + \sum_{i \in I_+ \cap J_+} c_i^* T_i = c_0^* + \sum_{i \in J_+} c_i^* T_i + \sum_{i \in I_- \cap J_0} c_i^* T_i \geq S(\widetilde{\mathbf{T}}).$$

This is valid due to the equality $I_- = (I_- \cap J_+) \cup (I_- \cap J_0)$. Note that $\widetilde{\mathbf{T}} \geq \widehat{\mathbf{T}}$ because $J_+ \subset J_+ \cup (I_- \cap J_0)$. Hence there holds $S(\widetilde{\mathbf{T}}) \geq S(\widehat{\mathbf{T}})$.

Now we prove that $\overline{h}(\widetilde{C}) \leq \overline{h}(C^*)$. Since we assume that $c_i^* \leq 0$, $i \in I_-$, then

$$\overline{h}(\widetilde{C}) = c_0^* + \sum_{i \in I_-} c_i^* T_i + \sum_{i \in I_+} c_i^* \overline{h}_i \leq$$

$$c_0^* + \sum_{i \in I_-} c_i^* \underline{h}_i + \sum_{i \in I_+} c_i^* \overline{h}_i = \overline{h}(C^*).$$

In sum, there holds $\mathbf{C} \geq 0$ and problem (8) is equivalent to problem (7), as was to be proved.

Theorem 3 implies that the upper prevision of a system depends only on upper previsions of primary gambles. In other words, the upper mean level of system performance depends only on upper mean levels of component performances.

5 Reliability of Systems

5.1 Coherent Systems with General Structure Functions

Theorem 1 implies that multistate and continuum systems can be considered as classical binary systems with the coherent structure function $S : \{0, T\}^n \to \{0, T\}$. This conclusion simplifies the calculation of the reliability measures of various systems. Below we will study the general rules for computing the lower and upper mean levels of system performance for arbitrary systems.

Suppose that a coherent structure S of a binary system has p minimal paths $P_1, \dots, P_p$ and k minimal cuts $K_1, \dots, K_k$. Then according to [15], the function S can be represented as

$$S(\mathbf{X}) = \max_{1 \leq j \leq p} \min_{i \in P_j} x_i = \min_{1 \leq j \leq k} \max_{i \in K_j} x_i.$$

Denote $N = \{1, \dots, n\}$.

Theorem 4. *Suppose* $\underline{h}_i(t)$, $i = 1, \ldots, n$, *are the lower mean levels of component performances. For computing the lower mean levels of system performance* $\underline{h}$, *the natural extension can be used in the following form:*

$$\underline{h} = \sup_{\mathbf{C} \geq 0, c_0} \left(c_0 + \sum_{i=1}^{n} c_i \underline{h}_i \right), \tag{10}$$

$$c_0 + \sum_{j \in N \setminus K_i} c_j T \leq 0, \ i = 1, \ldots, k,$$

$$c_0 + \sum_{i=1}^{n} c_i T \leq T, \ c_0 \leq 0.$$

Proof. Denote $I = \{i : x_i = 0\}$. According to Theorems 1 and 2, the system of constraints has the following form

$$c_0 + \sum_{j \in N \setminus I} c_j T \leq S(\mathbf{T}_{N \setminus I}) = \begin{cases} 0 \\ T \end{cases},$$

where $\mathbf{T}_{N \setminus I}$ is a vector with non-zero components T_j, $j \in N \setminus I$. If $I = N$, then $c_0 \leq 0$. Let $S(\mathbf{T}_{N \setminus I}) = T$. Then inequalities

$$c_0 + \sum_{j \in N \setminus I} c_j T \leq T, \ \forall I \neq \emptyset,$$

follow from the inequality $c_0 + \sum_{i=1}^{n} c_i T \leq T$. Let $S(\mathbf{T}_{N \setminus I}) = 0$. The system fails if all components belonging to a minimal cut set fail. This implies that $I \supset K_i$ or $I = K_i$. Let $I \supset K_i$. Then inequalities

$$c_0 + \sum_{j \in N \setminus I} c_j T \leq 0, \ \forall I \supset K_i,$$

follow from the inequality

$$c_0 + \sum_{j \in N \setminus K_i} c_j T \leq 0,$$

because $N \setminus K_i \subset N \setminus I$, as was to be proved.

Theorem 5. *Suppose* $\overline{h}_i(t)$, $i = 1, \ldots, n$, *are the upper mean levels of component performances. For computing the upper mean levels of system performance* $\overline{h}$, *the natural extension can be used in the following form:*

$$\overline{h} = \inf_{\mathbf{C} \geq 0, c_0} \left(c_0 + \sum_{i=1}^{n} c_i \overline{h}_i \right), \tag{11}$$

$$c_0 + \sum_{j \in P_i} c_j T \geq T, \ i = 1, \ldots, p, \ c_0 \geq 0.$$

Proof. Denote $I = \{i : x_i = T\}$. According to Theorems 1 and 3, the system of constraints has the following form

$$c_0 + \sum_{j \in I} c_j T \geq S(\mathbf{T}_I) = \begin{cases} 0 \\ T \end{cases},$$

where $\mathbf{T}_I$ is a vector with non-zero components T_j, $j \in I$. If $I = \emptyset$, then $c_0 \geq 0$. Let $S(\mathbf{T}_I) = 0$. Then inequalities $c_0 + \sum_{j \in I} c_j T \geq 0 \ \forall I \neq \emptyset$ follow from the inequality $c_0 \geq 0$. Let $S(\mathbf{T}_I) = T$. The system operates if all components belonging to a minimal path operate. This implies that $I \supset P_i$ or $I = P_i$. Let $I \supset P_i$. Then inequalities

$$c_0 + \sum_{j \in I} c_j T \geq T, \ \forall I \supset P_i,$$

follow from the inequality

$$c_0 + \sum_{j \in P_i} c_j T \geq T.$$

This completes the proof.

Theorems 4 and 5 reveal the interesting underlying relationships among the sets of constraints in the natural extension and minimal paths and cuts. Theorems allow us to develop the formal rules for constructing the system of constraints for arbitrary systems. Moreover, they reduce the number of constraints and essentially simplify the linear optimization problems. Their usage will be illustrated below.

One of the conventional ways for simplifying the reliability analysis of systems is the property of a decomposition. Let notation (A, χ) mean a coherent system with a structure function χ. Here A denotes a set of integers designating the components.

Definition 1. *A modular decomposition of a coherent system (C, ϕ) is a set of disjoint modules (A_1, χ_1),..., (A_m, χ_m) together with an organizing structure ψ:*

$$\begin{aligned} &\text{(a)} \ \ C = \textstyle\bigcup_{i=1}^m A_i, \ A_i \cap A_j = \emptyset \text{ for } i \neq j, \\ &\text{(b)} \ \ \phi(\mathbf{Z}) = \psi\left(\chi_1(Z^{A_1}), \ldots, \chi_m(Z^{A_m})\right), \end{aligned}$$

where Z^{A_i} denotes the vector with m_i elements $z_i \in \{0, T\}, i \in A_i$.

Theorem 6. *For a coherent system ϕ, suppose $(A_1, \chi_1), \ldots, (A_m, \chi_m)$ constitute one of its modular decompositions with organizing structure ψ, which has p minimal paths $P_1, \ldots, P_p$, k minimal cuts $K_1, \ldots, K_k$, and the upper and lower mean levels of subsystem performance $\overline{\mathbf{V}} = (\overline{v}_1, \ldots, \overline{v}_m)^{\mathrm{T}}$, $\underline{\mathbf{V}} = (\underline{v}_1, \ldots, \underline{v}_m)^{\mathrm{T}}$. Then the upper mean levels of system performance is determined as the solution of the following optimization problem:*

$$\overline{h} = \inf_{\mathbf{C} \geq 0, c_0} \left(c_0 + \mathbf{C}\overline{\mathbf{V}}\right), \tag{12}$$

subject to

$$c_0 + \mathbf{C}\mathbf{X} \geq \psi(\mathbf{X}) = \max_{1 \leq j \leq p} \min_{i \in P_j} x_i, \ \mathbf{X} = (x_1, \ldots, x_m)^{\mathrm{T}}.$$

The lower mean levels of system performance is determined as the solution of the following optimization problem:

$$\underline{h} = \sup_{\mathbf{C}\geq 0, c_0} \left(c_0 + \mathbf{C}\underline{\mathbf{V}}\right), \tag{13}$$

subject to

$$c_0 + \mathbf{C}\mathbf{X} \leq \psi(\mathbf{X}) = \max_{1\leq j\leq p} \min_{i\in P_j} x_i,\ \mathbf{X} = (x_1, \dots, x_m)^{\mathrm{T}}. \tag{14}$$

Proof. The vector $\overline{\mathbf{V}}$ can be obtained as the solution of m optimization problems

$$\overline{v}_i = \inf_{\mathbf{D}_i\geq 0, d_i} \left(d_i + \mathbf{D}_i\overline{h}(\mathbf{Y}^{A_i})\right),\ d_i + \mathbf{D}_i\mathbf{Y}^{A_i} \geq \chi(\mathbf{Y}^{A_i}) = x_i,\ i = 1, \dots, m, \tag{15}$$

where $\mathbf{D}_i$ and $\mathbf{Y}^{A_i}$ are vectors with m_i components; $\overline{h}(\mathbf{Y}^{A_i})$ is the vector of upper previsions of gambles $\mathbf{Y}^{A_i}$.

By substituting the vector $\overline{\mathbf{V}}$ into problem (12), we obtain

$$\overline{h} = \inf_{\mathbf{C}\geq 0, c_0} \left(c_0 + \sum_{i=1}^{m} c_i \inf_{\mathbf{D}_i\geq 0, d_i} \left(d_i + \mathbf{D}_i\overline{h}(\mathbf{Y}^{A_i})\right)\right).$$

Since $c_0 \geq 0$, $\mathbf{C} \geq 0$, $\mathbf{D}_i \geq 0$, $d_i \geq 0$, $i = 1, \dots, m$, and $A_i \cap A_j = \emptyset$ for $i \neq j$, then there holds

$$\overline{h} = \inf_{\mathbf{C}\geq 0, c_0} \inf_{\mathbf{D}_i\geq 0, d_i, i=1,\dots,m} \left(c_0 + \sum_{i=1}^{m} c_i d_i + \sum_{i=1}^{m} c_i\mathbf{D}_i\overline{h}(\mathbf{Y}^{A_i})\right).$$

Note that the set of previsions $\overline{h}(\mathbf{Y}^{A_i})$, $i = 1, \dots, m$, is the set of upper previsions for all the components of the system with the structure function ϕ.

Now consider constraints. Note that $x_i = \chi_i(\mathbf{Y}^{A_i})$. By substituting constraints (15) into (14), we obtain

$$c_0 + \sum_{i=1}^{m} c_i \left(d_i + \mathbf{D}_i\mathbf{Y}^{A_i}\right) = c_0 + \sum_{i=1}^{m} c_i d_i + \sum_{i=1}^{m} c_i\mathbf{D}_i\mathbf{Y}^{A_i}$$

$$\geq \psi(\chi_1(\mathbf{Y}^{A_1}), \dots, \chi_m(\mathbf{Y}^{A_m})) = \phi\left(\mathbf{Y}^{A_1}, \dots, \mathbf{Y}^{A_m}\right).$$

This completes the proof.

The case of the lower prevision can be proved in a similar way.

The possibility of the modular decomposition of a coherent system is the very important and useful property of reliability assessments because many complex systems can be constructed with typical systems such as series and parallel systems.

5.2 Series Systems

Definition 2. *A series structure is such that*

$$S(\mathbf{X}) = \min(x_1, x_2, \dots, x_n).$$

Theorem 7. *Let $\underline{h}_i$ and $\overline{h}_i$, $i = 1, \dots, n$, be the lower and upper mean levels of component i performance for a series system with the structure function $S(\mathbf{X}) = \min(x_1, \dots, x_n)$, $x_i \in L$. Then the lower and upper mean levels of system performance are defined as follows:*

$$\underline{h} = \max\left(0, \sum_{i=1}^{n} \underline{h}_i - (n-1)T\right), \tag{16}$$

$$\overline{h} = \min_{i=1,\dots,n} \overline{h}_i. \tag{17}$$

Proof. From Theorem 4, we obtain the optimization problem for computing the lower prevision

$$\underline{h} = \sup_{\mathbf{C}\geq 0, c_0} \left(c_0 + \sum_{i=1}^{n} c_i \underline{h}_i\right),$$

$$c_0 + \sum_{j=1, j\neq i}^{n} c_i T \leq 0,\ i = 1, \dots, n,\ c_0 + \sum_{i=1}^{n} c_i T \leq T,\ c_0 \leq 0.$$

From constraints we can write $0 \leq c_i \leq 1$, $i = 1, \dots, n$. This implies that $c_0 \leq -T(n-1)$ if $c_i = 1$ and $c_0 \leq 0$ if $c_i = 0$, $i = 1, \dots, n$. Then

$$\underline{h} = \max\left(-T(n-1) + \sum_{i=1}^{n} \overline{h}_i, 0\right).$$

This completes the proof of (16).

From Theorem 5, we obtain the optimization problem for computing the upper prevision

$$\overline{h} = \inf_{\mathbf{C}\geq 0, c_0} \left(c_0 + \sum_{i=1}^{n} c_i \overline{h}_i\right),$$

$$c_0 + \sum_{i=1}^{n} c_i T \geq T,\ c_0 \geq 0.$$

We will find solutions to the problem at corner points of the feasible region. There are two sets of such the points: (1) $c_0 = 0$, $c_k = 1$, $c_i = 0$, $i \neq k$; (2) $c_0 = T$, $c_i = 0$, $i = 1, \dots, n$. If we consider the first solution, then $\overline{h} = \min_{i=1,\dots,n} \overline{h}_i$. If we consider the second solution, then $\overline{h} = T$. Note that $\overline{h}_i \leq T$. With this inequality we arrive at (17).

5.3 Parallel Systems

Definition 3. *A parallel structure is such that*

$$S(\mathbf{X}) = \max(x_1, x_2, \dots, x_n).$$

Theorem 8. *Let $\underline{h}_i$ and $\overline{h}_i$, $i = 1, \dots, n$, be the lower and upper mean levels of component i performance for a parallel system with the structure function $S(\mathbf{X}) = \max(x_1, \dots, x_n)$, $x_i \in L$. Then the lower and upper mean levels of system performance are defined as follows:*

$$\underline{h} = \max_{i=1,\dots,n} \underline{h}_i, \tag{18}$$

$$\overline{h} = \min\left(\sum_{i=1}^{n} \overline{h}_i, T\right). \tag{19}$$

Proof. From Theorem 4, we obtain the optimization problem for computing the lower prevision

$$\underline{h} = \sup_{\mathbf{C}\geq 0, c_0} \left(c_0 + \sum_{i=1}^{n} c_i \underline{h}_i\right),$$

$$c_0 + \sum_{i=1}^{n} c_i T \leq T,\ c_0 \leq 0.$$

We will find solutions to the problem at corner points of the feasible region. There are two sets of such the points: (1) $c_0 = 0$, $c_k = 1$, $c_i = 0$, $i \neq k$; (2) $c_0 = 0$, $c_i = 0$, $i = 1, ..., n$. If we consider the first solution, then $\underline{h} = \max_{i=1,\dots,n} \underline{h}_i$. If we consider the second solution, then $\underline{h} = 0$. Note that $\underline{h}_i \geq 0$. With this inequality we arrive at (18).

From Theorem 5, we obtain the optimization problem for computing the upper prevision

$$\overline{h} = \inf_{\mathbf{C}\geq 0, c_0} \left(c_0 + \sum_{i=1}^{n} c_i \overline{h}_i\right),$$

$$c_0 + c_i T \geq T,\ i = 1, \dots, n,\ c_0 \geq 0.$$

It follows from the above inequalities that $c_i \geq \max(0, 1 - c_0/T)$. Therefore, we can write the following optimization problem:

$$\overline{h} = c_0 + \sum_{i=1}^{n} \overline{h}_i \max\left(0, 1 - \frac{c_0}{T}\right) \to \min_{c_0 \geq 0}.$$

If $0 \leq c_0 \leq T$, then

$$\overline{h} = c_0 + \sum_{i=1}^{n} \overline{h}_i \left(1 - \frac{c_0}{T}\right) = \sum_{i=1}^{n} \overline{h}_i + c_0 \left(1 - \sum_{i=1}^{n} \frac{\overline{h}_i}{T}\right).$$

Minimum of $\overline{h}$ is achieved by $c_0 = 0$ or $c_0 = T$. In the first case, there holds $\overline{h} = \sum_{i=1}^{n} \overline{h}_i$, in the second case, we have $\overline{h} = T$. If $T \leq c_0$, then $\overline{h} = c_0 = T$. In sum, we obtain

$$\overline{h} = \min \left(\sum_{i=1}^{n} \overline{h}_i, T \right).$$

This completes the proof of (19).

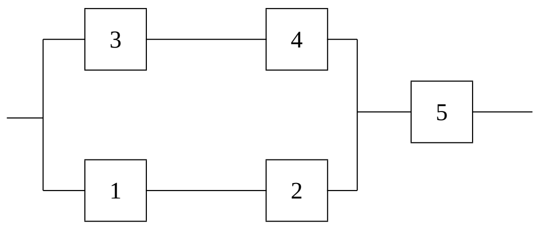

Fig. 1. A series-parallel system with two states

Example 3. Let us consider a system given in Fig. 1 whose components may be in two states: perfect functioning (A) and complete failure (B). To express the uncertainty about the reliability of components, experts make the judgements:

- failures of components 1 and 3 are very unlikely.
- failure of component 2 has very low probability.
- functioning of component 4 is extremely unlikely.
- functioning of component 5 is probable.

Let us translate the judgements in natural language into lower and upper probabilities by using Walley's paper [7].

- Components 1 and 3: $\overline{P}(B_1) = \overline{P}(B_3) \leq 0.25$.
- Component 2: $\overline{P}(B_2) \leq 0.1$.
- Component 4: $\overline{P}(A_4) \leq 0.02$.
- Component 5: $\underline{P}(A_5) \geq 0.5$.

Here $\underline{P}(A_i), \underline{P}(B_i)$ and $\overline{P}(A_i), \overline{P}(B_i)$ denote lower and upper probabilities that the ith component functions or fails, respectively. From the above assessments, we can write the vacuous probabilities: $\underline{P}(B_1) = \underline{P}(B_3) \geq 0$, $\underline{P}(A_4) = 0$, $\underline{P}(B_2) \geq 0$, $\overline{P}(A_5) \leq 1$. By using equalities $\overline{P}(A_i) = 1 - \underline{P}(B_i)$ and $\underline{P}(A_i) = 1 - \overline{P}(B_i)$, we can compute the following probabilities:

- Components 1 and 3: $\overline{P}(A_1) \leq 1$, $\underline{P}(A_1) \geq 0.75$, $\overline{P}(A_3) \leq 1$, $\underline{P}(A_3) \geq 0.75$.
- Component 2: $\overline{P}(A_2) \leq 1$, $\underline{P}(A_2) \geq 0.9$.

- Component 4: $\underline{P}(B_4) \geq 0.98$, $\overline{P}(B_4) \leq 1$.
- Component 5: $\underline{P}(B_5) \geq 0$, $\overline{P}(B_5) \leq 0.5$.

By using the expressions for upper and lower previsions of series and parallel systems, we can obtain lower and upper probabilities that the system is functioning:

$$\underline{h} = \max\left(0, \underline{h}_1 + \underline{h}_2 + \underline{h}_5 - 2, \underline{h}_3 + \underline{h}_4 + \underline{h}_5 - 2\right)$$
$$= \max\left(0, 0.75 + 0.9 + 0,5 - 2, 0.75 + 0 + 0.5 - 2\right) = 0.15$$
$$\overline{h} = \min\left(\overline{h}_5, \min(\overline{h}_1, \overline{h}_2) + \min(\overline{h}_3, \overline{h}_4)\right)$$
$$= \min\left(1, \min(1,1) + \min(1, 0.02)\right) = 1.$$

Here $\underline{h}_i = \underline{P}(A_i)$ and $\overline{h}_i = \overline{P}(A_i)$.

Example 4. Consider the performance level of a tube which is defined by sizes of cracks on the bore and surface of it. Examination of the bore of the tube by experts revealed a crack with the mean size 3–4 mm. Examination of the surface of the tube revealed a crack with the mean size 2–2.5 mm. It is supposed that the maximal size of cracks is equal to 8 mm and the tube is destructed if both cracks have the size 8 mm. The tube can be regarded as a continuum system with the structure function $S(x_1, x_2) \in L = [0, 8]$ and $x_1, x_2 \in L = [0, 8]$. Moreover, it follows from the description of the tube that $S(x_1, x_2) = \max(x_1, x_2)$. Here $\underline{h}_1 = 3$, $\overline{h}_1 = 4$, $\underline{h}_2 = 2$, $\overline{h}_2 = 2.5$. Now we can obtain the performance levels of the tube as follows:

$$\underline{h} = \max\left(\underline{h}_1, \underline{h}_2\right) = 3,$$
$$\overline{h} = \min\left(\overline{h}_1 + \overline{h}_2, T\right) = 6.5.$$

In spite of simplicity of linear optimization problems (10), (11), the reliability analysis of systems with a large number of components is the difficult computational problem. Therefore, the explicit expressions for lower and upper mean levels of system performance are desirable.

Theorem 9. *Suppose that a coherent structure S of a binary system has p minimal paths $P_1, \ldots, P_p$ containing $m_1, \ldots, m_p$ components, respectively. Let $\underline{h}_i$, $i = 1, \ldots, n$, be the lower mean level of component i performance. Then the lower mean level of system performance is determined as*

$$\underline{h} = \max_{1 \leq j \leq p} \max\left(0, \sum_{i \in P_j} \underline{h}_i - (m_j - 1)T\right).$$

Proof. First we have to prove that the assumption of independence for components of a parallel system is not required when we compute the lower mean level of parallel system performance. Suppose that lth and mth components of a parallel system consisting of n components are dependent, i.e. there holds $x_l = x_m$. Write the system of constraints to the optimization problem corresponding to the lower prevision

$$c_0 + \sum_{i \neq l, i \neq m} c_i x_i + c_l \cdot 0 + c_m \cdot 0 \leq \max_{i \neq l, i \neq m} x_i,$$

$$c_0 + \sum_{i \neq l, i \neq m} c_i x_i + c_l \cdot T + c_m \cdot T \leq T.$$

If $x_i = 0 \ \forall i \neq l, i \neq m$, then it follows from the first inequality that $c_0 \leq 0$. If $\exists i \neq l, i \neq m \ x_i = T$, then the obtained inequality follows from the inequality $c_0 + \sum_{i=1}^{n} c_i T \leq T$. In sum, we obtain constraints which coincide with constraints for the parallel systems with independent components (see proof for Theorem 8).

Now we can represent the system as a parallel connection of independent minimal paths, where each minimal path is a series system.

Theorem 10. *Suppose that a coherent structure S of a binary system has k minimal cut sets $K_1, \dots, K_k$. Let $\overline{h}_i$, $i = 1, \dots, n$, be the upper mean level of component i performance. Then the upper mean level of system performance is determined as*

$$\overline{h} = \min_{1 \leq j \leq k} \min \left(\sum_{i \in K_j} \overline{h}_i, T \right).$$

Proof. Similar to the proof for Theorem 9.

Example 5. Consider a system with $n = 5$ component, whose graphical representation is depicted in Fig. 2. For this system, we have $p = 4$ minimal paths $P_1 = \{1, 4\}$, $P_2 = \{2, 5\}$, $P_3 = \{1, 3, 5\}$, $P_4 = \{2, 3, 4\}$ and $k = 4$ minimal cuts $K_1 = \{1, 2\}$, $K_2 = \{4, 5\}$, $K_3 = \{1, 3, 5\}$, $K_4 = \{2, 3, 4\}$. Let $\underline{h}_i$ and $\overline{h}_i$, $i = 1, \dots, 5$, be the lower and upper mean levels of component i performance. By using Theorems 9 and 10, we obtain the lower and upper mean levels of system performance as follows:

$$\underline{h} = \max\left(0, \underline{h}_1 + \underline{h}_4 - T, \ \underline{h}_2 + \underline{h}_5 - T, \ \underline{h}_1 + \underline{h}_3 + \underline{h}_5 - 2T, \ \underline{h}_2 + \underline{h}_3 + \underline{h}_4 - 2T\right),$$

$$\overline{h} = \min\left(T, \underline{h}_1 + \underline{h}_2, \ \underline{h}_4 + \underline{h}_5, \ \underline{h}_1 + \underline{h}_3 + \underline{h}_5, \ \underline{h}_2 + \underline{h}_3 + \underline{h}_4\right).$$

6 Conclusion

In this paper we have shown that the reliability of systems with the imprecise structure functions can be analyzed by means of the theory of imprecise probabilities. We have illustrated that the reliability assessments can be obtained even by extremely restricted information. With the mean levels of component performance as information on the functioning of components, the following conclusions have been made:

- The multistate and continuum systems can be regarded as two-state systems. In other words, we take into account the bounds of states.
- There is the underlying relationships among the natural extension and minimal paths and cuts.
- The property of the modular decomposition of coherent systems is valid.
- For the series-parallel systems, bounds of mean levels of system performance can be obtained in the explicit form.

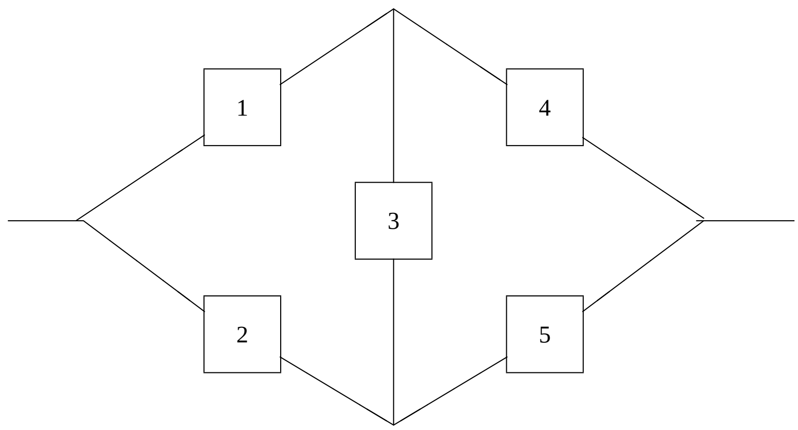

Fig. 2. A system with two states

– For arbitrary systems with known minimal paths and cuts, bounds of mean levels of system performance can be obtained in the explicit form.

It should be noted that obtained results have to be considered as an attempt to develop the general reliability theory. At the same time, results of the paper have the strong mathematical sense and can be used in practice. We can expect that the general reliability theory will become a powerful tool for reliability analyzing and will play a dominant role in developing new reliability models.

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References

1. K. Y. Cai, C. Y. Wen, M.L. Zhang. Fuzzy variables as a basis for a theory of fuzzy reliability in the possibility context, *Fuzzy Sets and Systems* **42**, 145–172, 1991.
2. K. Y. Cai. System failure engineering and fuzzy methodology. An introductory overview, *Fuzzy Sets and Systems* **83**(2), 113–133, 1996.
3. K. Y. Cai. *Introduction to Fuzzy Reliability*. Kluwer Academic Publishers, Boston, 1996.
4. G. de Cooman. On modeling possibilistic uncertainty in two-state reliability theory. *Fuzzy Sets and Systems* **83**(2), 215–238, 1996.
5. L. V. Utkin. Fuzzy reliability of repairable systems in the possibility context, *Microelectronics and Reliability* **34**(12), 1865–1876, 1994.

6. L.V. Utkin, S.V. Gurov. A general formal approach for fuzzy reliability analysis in the possibility context, *Fuzzy Sets and Systems* **83**, 203–213, 1996.
7. P. Walley. Measures of uncertainty in expert systems, *Artificial Intelligence* **83**, 1–58, 1996.
8. V. P. Kuznetsov. *Interval Statistical Models*, Radio and Communication, Moscow, 1991 (in Russian).
9. P. Walley. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall: London, 1991.
10. D. Dubois, H. Prade. When upper probabilities are possibility measures, *Fuzzy Sets and Systems* **49**, 65–74, 1992.
11. G. de Cooman. Describing linguistic information in a behavioural framework: Possible or not? In: J. Albus, A. Meystel, R. Quintero (eds.), *Intelligent Systems: A Semiotic Perspective (Proceedings of the 1996 International Multidisciplinary Conference)*, Gaithersburg, MD, 20–23 October 1996, 1996, pp. 141–150.
12. S. Moral, N. Wilson. Revision rules for convex sets of probabilities. In: G. Coletti, D. Dubois, R. Scozzafava (eds.), *Mathematical Models for Handling Partial Knowledge in Artificial Intelligence*, Plenum Press: New York, 1995, pp. 113–128.
13. J. Montero, J. Tejada, J. Yanez. General structure functions, *Kybernetes* **23**(3), 10–19, 1994.
14. R. E. Barlow, A. S. Wu. Coherent systems with multistate components, *Math. Ops. Res.* **3**, 275–281, 1978.
15. R. E. Barlow, F. Proschan. *Mathematical Theory of Reliability*. Wiley: New York, 1965.

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