Load-share reliability models with the piecewise constant load

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Abstract: Methods for reliability analysis of the load-share models are studied in the paper. In load-share systems, component or system failure rates depend on the working state of the other components in the system or depend on changable conditions of system functioning. A special type of the load, namely, the piecewise constant load is investigated. The main assumption used in the paper is the so-called "condition of the residual lifetime conservation" of the system, which is equivalent to the condition of continuity of the reliability (survivor) function. Rather simple recurrent expressions are obtained for computing reliability measures. Various numerical examples illustrate the proposed models.

Keywords: reliability; system load-share model; failure rate; survivor function; exponential distribution; mean time to failure.

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1 Introduction

A system after a certain time of usage is often affected some changes which impact on the system lifetime and the system reliability behavior. These changes may have the inherent as well as external reasons. The external changes are usually concerned with possible changes of conditions of the system usage, for instance, changes of temperature, moistness, electromagnetic noise, hostile environment, vibration, mechanical shocks, etc.

Another reason of the changes is some stochastic dependency among the system's units, which means that the reliability behavior of some units in the system depends on the state (working or failed) of other units. This dependency can be modelled by means of the so-called *load-share models*. According to load-share models, failure rates of units in a system depend on states of other units of the system. A crucial point of the models is the rule that governs how failure rates of units change after failures of other system units.

One of the pioneering works devoted to load-share models applied in the textile industry was proposed by Daniels Daniels (1945). Last decades the load-share models were the research subjects of many authors, for instance, Coleman (1957); Ross (1984); Durham et al. (1995); Lynch (1999); Kim and Kvam (2004); Stefanescu and Turnbull (2006). It should be noted that the above works are a very small part of a huge number of papers devoted to load-share models due to their importance in many applications. A nice review and many application examples of the load-share models were provided by Kvam and Pena (2005). Kvam and Pena illustrate that the load-share models can give more realistic models in various fields, in particular, in reliability analysis. For instance, the load-sharing models generalize the dynamic models for software reliability, where the basic problem is to assume that an unknown number of faults exist in the system (i.e., software). After a fixed time, some faults are detected, and the number of remaining faults is to be estimated. The load-share model represents a more flexible and realistic method of predicting the detection of faults by acknowledging the dynamic nature of fault detection when some faults have already been found. For instance, in problems where the number of software bugs is relatively small, the discovery of a major defect can help conceal or reveal other existing bugs in the software.

An interesting application example of the load-share models in studying jet engines functioning can be found in Shao and Lamberson (1991). Another very illustrative application field of the load-sharing models is civil engineering. Kvam and Pena (2005) describe a large structure supported by welded joints. The structure fails only after a series of supporting joints fail. The failure of one or two welded joints in a bridge support, for instance, might cause the stress on remaining joints to increase, thus causing earlier subsequent failures. Static reliability models fail to consider the changing stress in this setting, which constitutes a load-sharing model.

It should be noted that most works devoted to the load-sharing models consider their statistical aspects, i.e., they determine the parameters of changes under various conditions of functioning from statistical data. In the proposed paper, we study how the reliability measures change under some special types of the load conditions. The main assumption used in the paper is the so-called "condition of the residual lifetime conservation" of the system, which is equivalent to the condition of continuity of the cumulative distribution function of time to failure or reliability (survivor) function. For simplifying some expression by computing reliability measures, we also assume that the load increases in k times as the system failure rate increases in k times.

Let $P(t)$ be the reliability of a system under normal (or initial) conditions of its usage. The system reliability behavior or its functioning under the piecewise constant load can be modeled in the following way. The load is constant during certain (non-random) periods of time. Stepwise changes of the load occur at some discrete time points. These stepwise changes cause some changes of the system reliability behavior. It is assumed that the system reliability increases in case of the reduced load and vice versa. The idle states of a system (when the system is not working and not failed) can be regarded as the zero-valued load.

The main problem of the system analyzing is to determine the reliability of the system denoted by $P_c(t)$ by taking into account the stepwise changes of the load. We will also use the load factor as a measure of the load.

In order to give an example of the load measure, we consider a parallel system consisting of n units. It is supposed here that $n-1$ units are redundant. After failure of a unit, other units are under the increased load which is measured by the so-called *load factor* denoted k that is $k = n/(n-1)$ for the considered parallel system. After failure of the second unit, the load increases again and the load factor becomes to be $k = n/(n-2)$. By continuing the consideration of failures, we can say that the load on the last single working unit increases n times after failures of $n-1$ units and the load factor is now $k = n$. A simple analogy of the indicated behavior of the load is the electric current in a parallel circuit. If the parallel n-unit circuit resistance is R, then the resistance of the same circuit with $n-m$ working units increases and is $R_{n-m} = Rn/(n-m) = kR$.

If failures of the 1-st, 2-nd, ..., $(n-1)$-th units occur at time moments $t_1 < t_2 < ... < t_{n-1}$, then the function of the load factor under condition that its initial value is 1 has the following piecewise constant form:

$$k(t) = \begin{cases} 1, & \text{if } t < t_1, \\ n/(n-1), & \text{if } t_1 \leq t < t_2, \\ n/(n-2), & \text{if } t_2 \leq t < t_3, \\ & ... \\ n, & \text{if } t \geq t_{n-1}. \end{cases}$$

The time moments of the load changes $t_1, t_2, ..., t_{n-1}$ can be determinate or random. Therefore, the system behavior after the load changes a priori may be unknown because it depends on the time moments of changes. From the mathematical point of view, the above means that the system lifetime distribution changes under the load.

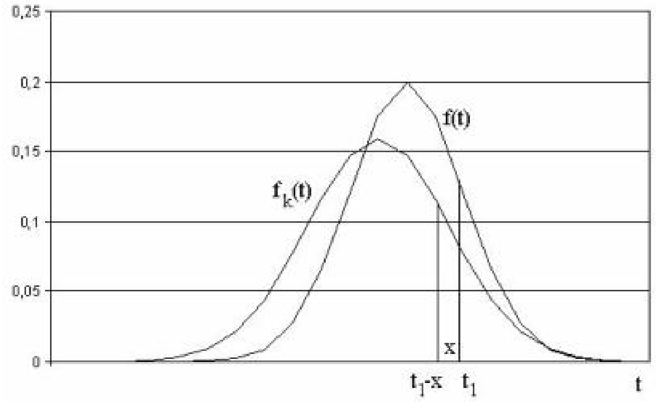

Figure 1 Probability density functions f and f_k

We investigate in the paper how types of probability distributions influence on the reliability measures of systems with different loads. The measures for analyzing are the survivor or reliability function and the mean time to failure (MTTF).

2 The single change of the load

First, we study a case when the load on a system changes only once. Let t_1 be a time of the load change, $P(t)$ be the reliability of the system under initial conditions of working, $P_k(t)$ be the reliability of the system under condition that the load has changed and became to be k. At that, the value of k may be larger or smaller than 1. The first case corresponds to the increased load, the second one corresponds to the decreased load. We assume that $k = 1$ under initial conditions of the system usage. This implies that k can be called the relative load factor.

The main idea underlying the computation of the reliability measure $P_c(t)$ after the load changes is the so-called "condition of the residual lifetime conservation" of the system. Namely, it can be written as

$$P_c(t) = \begin{cases} P(t), & \text{if } t < t_1, \\ P_k(t - x), & \text{if } t \geq t_1, \end{cases} \tag{1}$$

where the value of the "shift" x is chosen such that the area under the probability density function $f(t) = -P'(t)$ in interval $[t_1, +\infty)$ would be equal to the area under the probability density function $f_k(t) = -P_k'(t)$ in interval $[t_1 - x, +\infty)$ (see Fig. 1).

The above condition is equivalent to the condition of continuity of the cumulative distribution function of time to failure. It is obvious that the following continuity condition is valid in this case:

$$P_k(t_1 - x) = P(t_1). \tag{2}$$

Hence we can find the value of x as

$$x = t_1 - P_k^{-1}(P(t_1)),$$

and the reliability measure as

$$P_c(t) = \begin{cases} P(t), & \text{if } t < t_1, \\ P_k\left(t - t_1 + P_k^{-1}(P(t_1))\right), & \text{if } t \geq t_1. \end{cases}$$

Here P_k^{-1} is the inverse function.

Note that the value of the shift x may be positive as well as negative. The shift x is positive when the inequality $k > 1$ is valid. The negative value of x means that there holds $k < 1$.

It is interesting to study the limited values of the load factor. The case $k \to +0$ means that the system is in the idle state after time t_1. In this case, there holds $P_c(t) = P_c(t_1)$ by $t \geq t_1$. The case $k \to +\infty$ corresponds to the instantaneous failure at time t_1. Then $P_c(t) = 0$ by $t \geq t_1$.

The replacement of the function $P(t)$ by the function $P_k(t)$ at time t_1 of the load change can be carried out by several ways. Let us explain why the expression (1) is taken for computing the survivor probability distribution function $P_k(t)$.

1. It is obvious that the function $P_c(t)$ has to coincide with the function $P(t)$ before time t_1. In order to continue the graph of $P_c(t)$, we could use the parallel shift of the graph of $P_k(t)$, for instance, up or down, to the right or to the left, or at an angle. At that, the shifted graph of $P_k(t)$ should pass through the point $(t_1, P(t_1))$. However, this shift does not violate the residual part of the graph $P_k(t)$ only in the case when the shift is carried out in parallel to the abscissa axis, i.e., to the right when the load increases, and to the left when the load decreases.

2. For the failure aftereffect when $k > 1$, there exist boundary functions $P_c(t)$ caused by the lack or availability of the system "memory". The proof of this fact is given in Appendix A.

3. The "memory" is absent for the exponential distribution. Therefore, the reliability function coincides with its upper bound. The proof is given in Appendix B.

4. The case when the system shuts off (or activates after its idle state) in interval $(t_1, +\infty)$ corresponds to the load factor $k = 1$ for $t < t_1$ and $k = 0$ for $t \geq t_1$. Indeed, it follows from (2) that $x \to -\infty$ and we have $P_c(t) = P(t_1)$ for $t \geq t_1$. This means that the reliability of the system is constant in the interval of the system idle state.

3 Analysis of the reliability functions

Let us study an important question how the function $P_k(t)$ depends on the load factor k. It should be noted that this problem has to be solved during analyzing the system lifetime distribution. The lifetime distribution is a function of time and it depends on external conditions of the system working, in particular, on the load. Therefore, the probability $P_k(t)$ is generally a function of two variables $P(t, k)$, where $t \geq 0$, $k \geq 0$. When the load factor is $k = 1$, we have "natural" conditions of the system working. When $k > 1$, we have "stressed" conditions. When $k < 1$, the system has reduced load conditions, for instance, a cold-standby system. In order to obtain the function $P_k(t)$, we have to realize a large number of reliability tests under different load conditions.

Rather simple expressions for $P_k(t)$ can be obtained if we link the load factor k and the system failure rate. In particular, suppose that the load increases in k times as the system failure rate increases in k times. Then there holds

$$P_k(t) = P^k(t). \tag{3}$$

Indeed, the above follows from the following equality:

$$P(t) = e^{-\Lambda(t)} = e^{-\int_0^t \lambda(x)dx}.$$

Here $\lambda(x)$ is the failure rate; $\Lambda(t)$ is the cumulative failure rate. Hence

$$P_k(t) = e^{-\int_0^t k\lambda(x)dx} = e^{-k\Lambda(t)} = P^k(t).$$

Equality (3) is the basis for the next calculations.

Let us consider an interesting case of a system with the oscillating load when the load factor step-wise increases and then step-wise decreases to the initial state. We also suppose that the system starts working at time t_1 under conditions of the increased load, but it returns to initial conditions at time t_2. Then the reliability function is

$$P_c(t) = \begin{cases} P(t), & \text{if } t < t_1, \\ P_k(t-x), & \text{if } t_1 \leq t < t_2, \\ P(t-y), & \text{if } t \geq t_2. \end{cases} \tag{4}$$

Here the positive value of the shift x is determined from equality (2), the value of the shift y is negative and is determined from equality $P(t_2 - y) = P_k(t_2 - x)$, that follows from the "condition of the residual lifetime conservation".

Example 3.1: Suppose there holds $P(t) = e^{-\lambda t}$ under "normal"[1] conditions of the system working. Here $\lambda = 0.1$ h^{-1}. We also suppose that the load factor is of the form:

$$k(t) = \begin{cases} 1, \text{if } t < 4, \\ 2, \text{if } 4 \leq t < 9, \\ 1, \text{if } t \geq 9. \end{cases}$$

It can be seen that the load changes twice at time $t_1 = 4$ h and at time $t_2 = 9$ h. Let us find the system reliability $P_c(t)$. By using equalities (3) and (4), we find values of the shifts x and y. Since

$$e^{-2\lambda(4-x)} = e^{-\lambda 4},$$

then $x = 2$. Since

$$e^{-\lambda(9-y)} = e^{-2\lambda(9-2)},$$

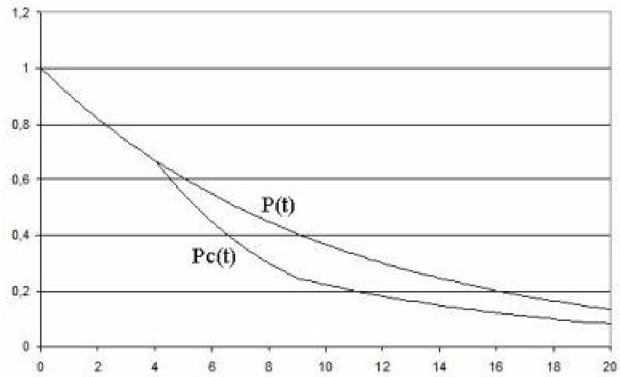

Figure 2 Functions $P(t)$ and $P_c(t)$ under different working conditions

then $y = -5$. Hence, the expression for $P_c(t)$ is of the form:

$$P_c(t) = \begin{cases} e^{-\lambda t}, & \text{if } t < 4, \\ e^{-2\lambda(t-2)}, & \text{if } 4 \le t < 9, \\ e^{-\lambda(t+5)}, & \text{if } t \ge 9. \end{cases}$$

Two curves are shown in Fig. 2. The 'upper' curve is the reliability function of the system $P(t)$ under "normal" working conditions when the load factor is 1. The 'lower' curve is the reliability function of the system $P_c(t)$ with the load factor $k(t)$.

4 Reliability analysis by the arbitrary piecewise constant load

Suppose now that the load changes at time moments $t_1, t_2, ..., t_m$ and the load factors become at these times to be $k_1, k_2, ..., k_m$, respectively. Then the reliability function of the system accounting the above load changes is

$$P_c(t) = \begin{cases} P_{k_0}(t), & \text{if } t < t_1, \\ P_{k_1}(t - x_1), & \text{if } t_1 \le t < t_2, \\ P_{k_2}(t - x_2), & \text{if } t_2 \le t < t_3, \\ ... & ... \\ P_{k_m}(t - x_m), & \text{if } t \ge t_m. \end{cases} \tag{5}$$

The value k_0 corresponds to the initial value of the load factor. For instance, if the system starts to work under "normal" conditions without an additional load, then there holds $k_0 = 1$. The function $P_k(t)$ is the probability that the system is working without failures by the load with factor k.

In order to satisfy "the condition of the residual lifetime conservation", the following equalities have to be valid:

$$P_{k_i}(t_i - x_i) = P_{k_{i-1}}(t_i - x_{i-1}), \ i = 1, 2, ..., m. \tag{6}$$

It is assumed here $x_0 = 0$. It can be seen that equality (2) is a special case of (6). Now we can determine all values of the shifts $x_1, x_2, ..., x_m$ from (6).

So, the lifetime taking into account the changeable load is a random variable which has the following additional parameters in comparison with the "normal" lifetime:

- $t_1, ..., t_m$ are the time moments when the load changes, $t_1 < t_2 < ... < t_m$;
- $k_1, ..., k_m$ are the corresponding load factors at time moments $t_1, ..., t_m$, $k_1, ..., k_m \geq 0$.

Let us consider the recurrent method for computing the function $P_c(t)$ under condition that equalities (3) are valid. We get from (5)

$$P_c(t) = P^{k_i}(t - x_i) \text{ by } t_i \leq t < t_{i+1}. \quad (7)$$

It follows from (6) that

$$P^{k_i}(t_i - x_i) = P^{k_{i-1}}(t_i - x_{i-1}).$$

Hence

$$x_i = t_i - P^{-1}\left(P^{\frac{k_{i-1}}{k_i}}(t_i - x_{i-1})\right), \ i = 1, 2, ..., m. \quad (8)$$

The reliability function of the system with the piecewise constant load is computed by using (7), where the shift parameters x_i satisfy recurrent equation (8).

It should be noted that the solution of (8) can be obtained in the explicit form when the lifetime is exponentially distributed. Assume $P(t) = e^{-\lambda t}$. Then we get from (8)

$$x_i = t_i - P^{-1}\left(e^{-\frac{k_{i-1}}{k_i}\lambda(t_i - x_{i-1})}\right).$$

Since $P^{-1}(y) = -\frac{1}{\lambda}\ln(y)$, then

$$x_i = t_i - \frac{1}{\lambda}\frac{k_{i-1}}{k_i}\lambda(t_i - x_{i-1}) = \frac{k_i - k_{i-1}}{k_i}t_i + \frac{k_{i-1}}{k_i}x_{i-1}.$$

Hence

$$k_i x_i = (k_i - k_{i-1})t_i + k_{i-1}x_{i-1}.$$

By consecutively summing the above equalities, we obtain

$$x_i = \frac{1}{k_i}\sum_{j=1}^{i}(k_j - k_{j-1})t_j.$$

Finally, there holds for the exponential distribution of the lifetime

$$P_c(t) = \begin{cases} \exp\left(-\lambda\left(k_i t - \tilde{x}_i\right)\right), & \text{if } t_i \leq t < t_{i+1}, \ \ i = 0, 1, ..., m-1, \\ \exp\left(-\lambda\left(k_m t - \tilde{x}_m\right)\right), \text{if } \ t \geq t_m. \end{cases}$$

where

$$\tilde{x}_i = \sum_{j=1}^{i} \left(k_j - k_{j-1}\right) t_j.$$

The MTTF T_c of the system with the piecewise constant load is of the form:

$$T_c = \int_0^\infty P_c(t)\, dt = \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} e^{-\lambda(k_i t - \tilde{x}_i)} dt + \int_{t_m}^{\infty} e^{-\lambda(k_m t - \tilde{x}_m)} dt.$$

After simple modification, we have

$$T_c = \sum_{i=0}^{m-1} \frac{1}{\lambda k_i} \left(e^{-\lambda(k_i t_i - \tilde{x}_i)} - e^{-\lambda(k_i t_{i+1} - \tilde{x}_i)} \right) + \frac{1}{\lambda k_m} e^{-\lambda(k_m t_m - \tilde{x}_m)}.$$

One can see from (7)-(8) that it is difficult to get rather simple expressions for $P_c(t)$ when the probability distributions of the system lifetime differ from the exponential one. However, it can be done by means of numerical methods and the corresponding specific software.

5 Numerical examples for various load conditions

First, we consider toy numerical examples illustrating some properties of systems under different load conditions.

Example 5.1: The lifetime of a system under the “normal” condition of working has the normal probability distribution with the expectation $m = 10$ h and the mean-square deviation $\sigma = 2$ h. The load increases at time $t = 9$ h and it is (a) $k = 3$, (b) $k = 10$. By using the numerical methods and the developed software program, we draw the reliability functions of the system. They are depicted in Fig. 3. The upper curve with number 1 corresponds to the “normal” condition by $k = 1$. The curve 2 is the reliability function $P_c(t)$ under the load with the factor $k = 3$. The lower curve with the number 3 corresponds to the reliability function $P_c(t)$ when $k = 10$. One can see from Fig. 3 that the functions $P_c(t)$ and $P(t)$ coincide before time $t_0 = 9$ h. However, after the time of increasing the load, the function $P_c(t)$ is below the function $P(t)$. The MTTF under “normal” conditions is 10 h. The MTTF under the load with $k = 3$ decreases till 9.62 h. The MTTF by $k = 10$ is 9.38 h.

Example 5.2: Suppose that the system lifetime under the “normal” condition of working has the gamma distribution with the expectation $m = 10$ h and the

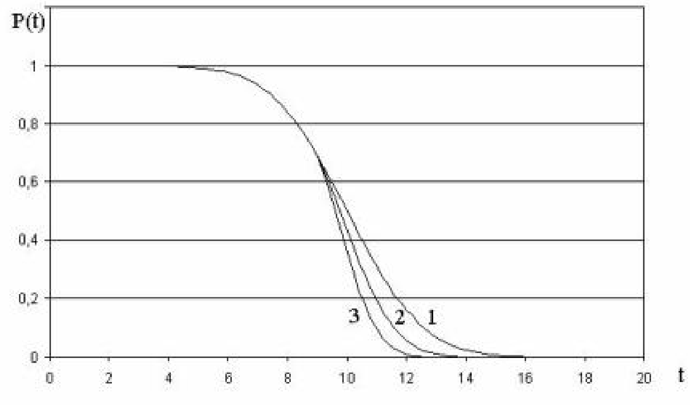

Figure 3 Functions $P_c(t)$ by different values of k

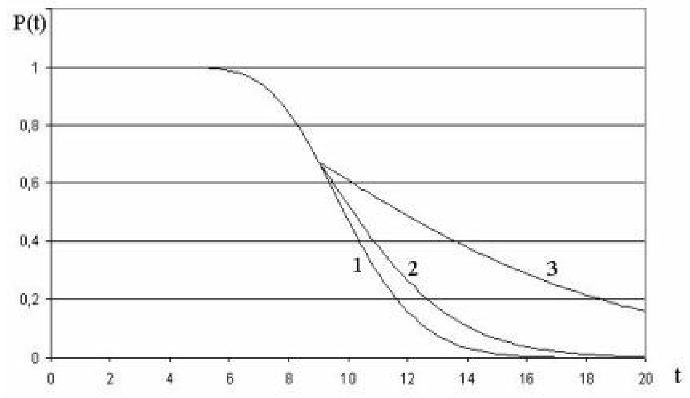

Figure 4 Functions $P_c(t)$ by different values of k

mean-square deviation $\sigma = 2$ h. The load decreases at time $t = 9$ h and it is (a) $k = 0.5$, (b) $k = 0.1$. By using the numerical methods and the developed software program, we draw the reliability functions depicted in Fig. 4. The lower curve with number 1 corresponds to the "normal" condition by $k = 1$. The curves 2 and 3 are the reliability functions $P_c(t)$ under the load with factors 0.5 and 0.1, respectively. The MTTF under "normal" conditions is 10 h. The MTTF under the load with $k = 0.5$ increases till 10.54 h. The load with $k = 0.1$ leads to the MTTF 12.77 h.

Example 5.3: Let us consider a case when there are 10 time moments of the load changes $t_i = i$, $i = 1, 2, ..., 10$. The load factor takes the values $k_i = 11/(11 - i)$, $i = 1, 2, ..., 10$, respectively. It is assumed that the expectation is $m = 10$ h and the mean-square deviation is $\sigma = 2$ h. The reliability function $P(t)$ under "normal" conditions of working and the reliability function $P_c(t)$ by assuming that the lifetime has the exponential distribution with the failure rate $\lambda = 0.1$

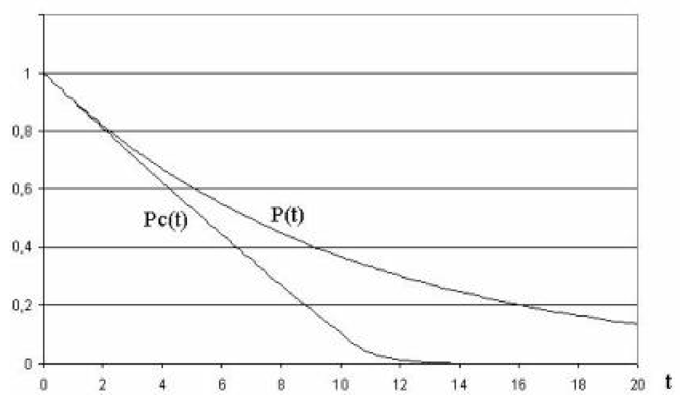

Figure 5 Functions $P\,(t)$ and $P_c\,(t)$ by the exponential distribution of the lifetime

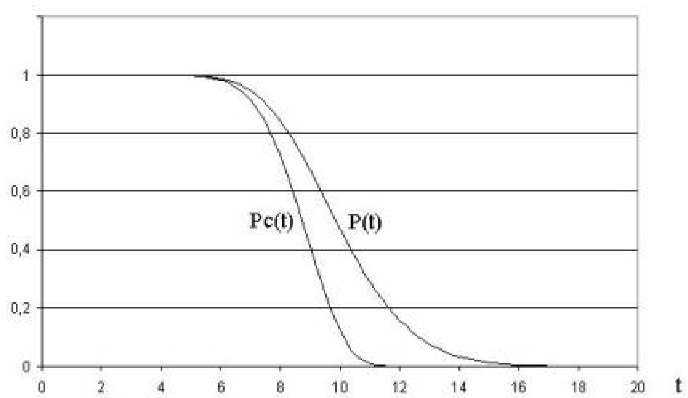

Figure 6 Functions $P\,(t)$ and $P_c\,(t)$ by the gamma distribution of the lifetime

h^{-1} are depicted in Fig. 5. Functions $P\,(t)$ and $P_c\,(t)$ by assuming the gamma distribution with the expectation $m = 10$ h and the mean-square deviation $\sigma = 2$ h are shown in Fig. 6. The similar forms have curves by assuming the normal distribution and Weibull distribution. Functions $P\,(t)$ and $P_c\,(t)$ by assuming the Rayleigh distribution with the parameter $\lambda = \pi/400$ h^{-2} are depicted in Fig. 7. Without taking into account the changes of the load, the MTTF is $m = 10$ h for all cases. By taking into account the changes, we get values of the MTTF shown in Table 1. It can be seen from Fig. 5, Fig. 6 and from Table 1 that the assumption of the exponential distribution provides the worst reliability.

Example 5.4: Suppose that the lifetime under "normal" conditions has the exponential distribution with the failure rate $\lambda = 0.1$ h^{-1}. The system is in the idle state in the following intervals (in hours): $[3; 5]$, $[9; 11]$, $[12; 15]$. The load factor in these intervals is 0.01. Let us find the reliability measures taking into account

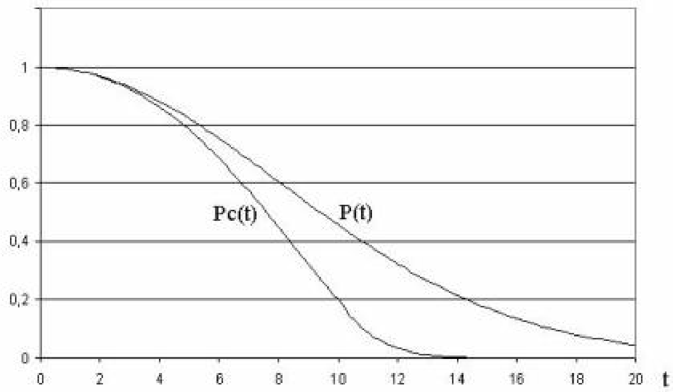

Figure 7 Functions $P(t)$ and $P_c(t)$ by the Rayleigh distribution of the lifetime

Table 1 Values of MTTF for different probability distributions

Distributions	MTTF, hours
Exponential	5.49
Gamma	8.66
Normal	8.93
Rayleigh	7.38
Weibull	9.00

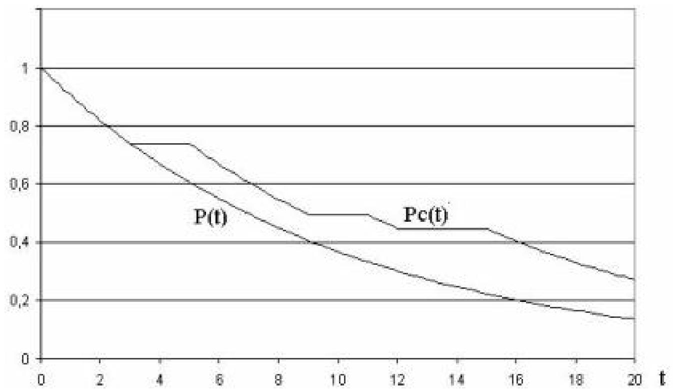

Figure 8 Functions $P(t)$ and $P_c(t)$ without and with intervals of the idle state

the idle states. By using the developed software program, we draw the reliability functions which are depicted in Fig. 8. In this case, it is possible to write the expressions for the reliability function in the explicit form. It can be seen from Fig. 8 that the idle states significantly impact on the system reliability. The MTTF without idle states is 10 h, and with idle states is 11.07 h.

6 Conclusion

Load-share reliability models with the piecewise constant load have been studied in the paper. The main feature of the models is applying the condition of the residual lifetime conservation to reliability analysis. The condition means that the cumulative distribution function of time to failure is continuous, i.e., it can not have any jumps. Of course, this assumption can be disputed. However, it adequately models the reliability behavior of many real systems.

Rather simple expressions have been obtained for computing the reliability measures by various types of the load change conditions. Moreover, the reliability function and the mean time to failure can be computed by using the simple recurrent algorithm when the load is changed at predefined time moments.

The various numerical examples have illustrated how the different conditions of system functioning can impact on its reliability.

Appendix A. Bounds for the reliability function of a system with load changes

Let t_1 be a time of the load change, X and $P(t)$ be the lifetime and the reliability of the system before time t_1 (before the load change), X_k and $P_k(t)$ be the lifetime and the reliability after time t_1 (after the load change). In other words, $P(t)$ is the reliability under "normal" conditions and $k=1$, $P_k(t)$ is the reliability under conditions that the load is equal to k.

We determine the survivor function of the lifetime X_c by taking into account the single change of the load at time t_1. By analyzing the random variable X_c, two cases can be studied:

1. The system becomes to be new after time t_1 and the period of its working before this time is not taken into account, i.e., the system does not have the "memory".
2. The lifetime before time t_1 is taken into account after this time, i.e., the system has the "memory".

The reliability function in the first case will be denoted $P_c^{(U)}(t)$, in the second case $P_c^{(L)}(t)$.

The lack of "memory"

Suppose that the lower value of the lifetime X_k is larger than the time moment of the load change t_1. Then we can write

$$X_c = \begin{cases} X, & \text{if } X < t_1, \\ t_1 + X_k, & \text{if } X \geq t_1. \end{cases}$$

By using the total probability formula, we get

$$P_c^{(U)}(t) = P(X < t_1, X \geq t) + P(X \geq t_1, t_1 + X_k \geq t).$$

If $t < t_1$, then $P_c^{(U)}(t) = P(t) - P(t_1) + P(t_1) = P(t)$. If $t \geq t_1$, then $P_c^{(U)}(t) = P(t_1) P_k(t - t_1)$. This implies that

$$P_c^{(U)}(t) = \begin{cases} P(t), & \text{if } t < t_1, \\ P(t_1) P_k(t - t_1), & \text{if } t \geq t_1. \end{cases} \tag{9}$$

The system with "memory"

Let us consider the second case. By assuming that the reliability "resource" of the lifetime X is spent starting from the beginning of the system working, but not from the time of the load change, we obtain

$$X_c = \begin{cases} X, & \text{if } X < t_1, \\ t_1, & \text{if } X \geq t_1, X_k < t_1, \\ X_k, & \text{if } X \geq t_1, X_k \geq t_1. \end{cases}$$

This implies that

$$\begin{aligned} P_c^{(L)}(t) &= P(X < t_1, X \geq t) \\ &+ P(X \geq t_1, X_k < t_1, t_1 \geq t) \\ &+ P(X \geq t_1, X_k \geq t_1, X_k \geq t). \end{aligned}$$

If $t < t_1$, then

$$P_c^{(L)}(t) = P(t) - P(t_1) + P(t_1)(1 - P_k(t_1)) + P(t_1) P_k(t_1) = P(t).$$

If $t \geq t_1$, then $P_c^{(L)}(t) = P(t_1) P_k(t)$. Hence

$$P_c^{(L)}(t) = \begin{cases} P(t), & \text{if } t < t_1, \\ P(t_1) P_k(t), & \text{if } t \geq t_1. \end{cases} \tag{10}$$

The function $P_k(t)$ is decreasing. Then it follows from (9)-(10) that

$$P_c^{(L)}(t) \leq P_c^{(U)}(t). \tag{11}$$

Inequality (11) means that the reliability of a system without "memory" is larger than the reliability of a system with "memory". This intuitively agrees with our understanding the reliability of systems with and without "memory".

If we do not know whether a system has "memory" or not, then the system reliability function $P_c(t)$ must satisfy the following two-sided inequality:

$$P_c^{(L)}(t) \leq P_c(t) \leq P_c^{(U)}(t). \tag{12}$$

Let us prove that the probability $P_c(t)$ determined from (1) satisfies the above two-sided inequality. It is sufficient to consider the case $t \geq t_1$. Let us write two natural assumptions:

1. The probability distribution $P_k(t)$ belongs to the class of all IFRA (increasing failure rate average) distributions Barlow and Proschan (1975),

i.e., the probability of the residual lifetime during time x is not larger than the probability of the lifetime for the same time

$$\frac{P_k(t+x)}{P_k(t)} \le P_k(x).$$

1. The load changes make the system to be less reliable, i.e., there holds

$$P_k(t) \le P(t).$$

The first assumption and equality (2) imply

$$\begin{aligned} P_k(t-x) &= P_k(t-t_1+t_1-x) \le P_k(t_1-x)\,P_k(t-t_1) \\ &= P(t_1)\,P_k(t-t_1). \end{aligned}$$

The above completes the proof of the upper bound.
The second assumption implies

$$P(t_1) = P_k(t_1-x) \le P(t_1-x).$$

Hence $x \ge 0$. Therefore, we get

$$P(t_1)\,P_k(t) \le P_k(t) \le P_k(t-x).$$

The above completes the proof of the lower bound.

So, we have two-sided bounds for the reliability of a system with aftereffect or with changes of the load.

Appendix B. The reliability function and its upper bound by the exponential distribution

Let us prove that $P_c(t) = P_k^{(U)}(t)$ when the lifetime is exponentially distributed. In this case, we have

$$P_c(t) = \begin{cases} \exp(-\lambda t), & \text{if } t < t_1, \\ \exp(-k\lambda(t-x)), & \text{if } t \ge t_1, \end{cases}$$

where x is such that

$$\exp(-k\lambda(t_1-x)) = \exp(-\lambda t_1).$$

Hence $x = t_1(k-1)/k$. Then

$$P_c(t) = \begin{cases} \exp(-\lambda t), & \text{if } t < t_1, \\ \exp(-\lambda(kt-(k-1)t_1)), & \text{if } t \ge t_1. \end{cases}$$

Let us consider the upper bound

$$P_c^{(U)}(t) = \begin{cases} P(t), & \text{if } t < t_1, \\ P(t_1)\,P_k(t-t_1), & \text{if } t \ge t_1. \end{cases}$$

By assuming the exponential distribution of the lifetime, we get

$$P_c^{(U)}(t) = \begin{cases} \exp(-\lambda t), & \text{if } t < t_1, \\ \exp(-\lambda(kt-(k-1)t_1)), & \text{if } t \ge t_1. \end{cases}$$

The above completes the proof.

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