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An inverse problem of the load-sharing system reliability analysis: Constructing the load function

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Abstract
The reliability analysis of the load-sharing systems under different behavior of the load has been provided in the paper. An inverse problem of determining the load function by having requirements concerning the system survivor function is solved. Simple and explicit expressions for computing the load function are provided under assumption of the Weibull probability distributions of time to failure are derived. A standby system consisting of two units under the changeable load is studied. The various numerical examples illustrate the proposed models.

Keywords
Reliability, load-sharing model, failure rate, survivor function, Weibull distribution, mean time to failure, standby system

1. Introduction

Many reliability models consider systems under unchangeable conditions of their functioning. Only a few models take into account these important conditions which impact on the real system reliability behavior. Changes of working conditions of systems are caused by changes of the load on the system and its units. The corresponding systems in some cases are called *load-sharing systems*. In these systems, unit failure rates depend on the working or failed states of other units in the system. If to consider the load-sharing model of a system consisting of some units, then failure of one unit can possibly put additional load on the surviving units and may reduce the residual time to failure of the system. On the other hand, the failure of a unit may release extra resources to other units and to increase the system time to failure. The first case leads to the increased failure rate. The second case corresponds to the reduced failure rate.

However, the unit failures can be regarded as one of possible reasons for load changes. Changes of the working conditions may often be caused by a complex of reasons, for instance, changes of temperature, moistness, electromagnetic noise, hostile environment, vibration, mechanical shocks, etc. A nice illustrative application example of a system with the changeable load is an electrical network whose load strongly depends on outdoor temperature and directly leads to the reliability behavior changes. For instance, deviations in outdoor temperature in Lapland (Finland) are large, the load increase may be up to 100% compared to the load at normal temperature (1). Valor et al. (2) indicate that the electricity

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load shows maximum values in winter and summer and minimum values in spring and autumn. As indicated by Tanrioven and Alam (3), studies that quantify power system reliability are often limited to constant transmission rates, covering two weather conditions, namely normal and adverse weather. More adequate models should be used in this case in order to take into account the load changes.

One of the pioneering works devoted to load-sharing models applied in the textile industry was proposed by Daniels (4) in 1945. Daniels originally adopted the load-sharing model to describe how the strain on yarn fibers increases as individual fibers within a bundle break. Last decades, many authors contribute to the load-sharing models. In particular, Bebbington et al. (5) considered the situation when the system units operate independently, but when any one of them fails, the load of the failed unit is instantaneously distributed among the working units. A semiparametric multivariate family of distributions which models the reliability behavior of a multi-unit load-sharing system was proposed by Deshpande et al. (6). It is important to point out that the authors (6) studied a case when the failure of one unit may release extra resources to the survivors, thus delaying the system failure. The load-sharing system consisting of independent units whose lifetimes have exponential distributions were investigated by Lee et al. (7), Durham and Lynch (8). The authors (7; 8) introduced the loading diagram to explicitly compute the survival distribution under a certain load-share rule. Lynch (9) showed that the joint probability distribution of failures of a load-sharing system under monotone load-sharing is the same as a conditional distribution of ordered independent gamma-type scale mixed random variables when the unit lifetimes are independent with a Weibull distribution. A system in which the failure rate of a unit at any time depends on the set of working units was considered by Ross (10). In this work, some special cases of repair were also studied.

Kim and Kvam (11) considered a multi-unit load-sharing system in which the failure rate of a given unit depends on the set of working units at any given time. The authors (11) assumed that the load-share rule is unknown and derived methods for statistical inference on load-share parameters. Kvam and Pena (12) proposed a semiparametric estimator for the load-share parameters in a load-share model, which is based on observing independent and identical parallel systems consisting of identical units. The effect of estimation of the load-share parameters was considered in the derivation of the limiting process. The authors (12) briefly described many potential application examples of the load-sharing systems where the proposed estimator can be applied, including software reliability, civil engineering, power plant safety, materials testing, population sampling, combat modeling.

Statistical parametric models for correlated failure time data which can be regarded as special case of the load-sharing models were proposed by Stefanescu and Turnbull (13). The authors (13) also discussed Bayesian methods for fitting these models to experimental data.

Volovoi (14) proposed the so-called universal failure model for reliability modelling the complex systems having a large number of units performing a common function. At that, the author (14) studied an interesting special case of the load-share model. He assumed that if failure occurs, the load is redistributed (shared) by the neighboring cells/nodes. This load redistribution has a characteristic neighborhood radius.

Yang and Younis (15) introduced a very flexible and powerful method of reliability analysis of systems with load sharing and damage accumulation. The method uses a semi-analytical Monte Carlo simulation technique for generating failures and takes into account the interaction effects between units of systems.

It is interesting to note that a lot of papers are devoted to reliability analysis of load-sharing m-out-of-n systems (16; 17; 18; 19; 20). This can be explained by the importance of m-out-of-n systems in modelling the reliability behavior of many real applications. In particular, Shao and Lamberson (18) analyzed the reliability and availability of an n-unit shared-load repairable m-out-of-n:G system with imperfect switching by applying a Markov model. An m-out-of-n:G system composed of statistically independent and identically distributed units with exponential lifetimes was investigated by Scheuer (17). Amari et al. (16) proposed a closed-form analytical solution for the reliability of tampered failure rate load-sharing m-out-of-n:G systems with identical units where all surviving units share the load equally. A load-sharing m-out-of-n: G system consisting of different components having exponential lifetime distributions was considered by Yinghui and Jing (19). Yun et al. (20) analyzed a linear and circular consecutive-m-out-of-n:F system consisting of identical units with exponential lifetime distributions and subjected to a total load that is equally shared by all the working units in the system. The authors (20) provided methods for choosing an optimal system configuration taking into account the cost parameters. A general closed-form expression for the lifetime reliability of load-sharing m-out-of-n hybrid redundant systems was proposed by Huang and Xu (21).

The load-sharing models also studied by Gurov and Utkin (22; 23). The authors considered how to compute the system reliability measures, for instance, the survivor function (SF), and how loads of different kinds impact on the reliability measures of a system. According to these models, the initial SF $P(t)$ of a system may be changed under increasing or decreasing load. Let $P_c(t)$ be the SF taking into account the load function (LF) $L(t)$ which can be regarded as a function of time. Most works devoted to load-sharing models, including the works proposed by Gurov and Utkin (22; 23), study how to compute the SF $P_c(t)$ of the system under the changeable load by having the initial SF $P(t)$ and the LF $L(t)$.

However, it is interesting to consider an inverse problem. How to determine the load function of a load-sharing system in order to provide a required level of reliability? The problem can be written in a more formal form. Suppose that a system having the SF $P\left(t\right)$ is under the additional load. As a result, the SF is changed and becomes to be $P_c\left(t\right)$. Our aim is to determine the function $L(t)$ which transforms $P\left(t\right)$ into $P_c\left(t\right)$ in order to achieve the required values of $P_c\left(t\right)$.

This problem has a connection with the following problems:

1. We analyze a system whose reliability is measured by a known SF $P\left(t\right)$. What is the additional load which can be shared by the system under condition that its SF $P_c\left(t\right)$ does not exceed a predefined threshold? Since we say about the additional load, then there holds $L\left(t\right) > 1$.
2. We analyze a system whose reliability is less than a predefined threshold. How to reduce the load to provide the reliability larger than a predefined threshold? Here $L\left(t\right) < 1$.

The inverse problem is important from theoretical as well as practical points of view. Therefore, we propose a general method for constructing the LF $L(t)$. Then we investigate special cases of systems, in particular, by assuming that times to failure are governed by the Weibull probability distributions. These special cases give us simple and explicit expressions for computing the LF under different conditions.

The paper is organized as follows. Section 2 provides a brief introduction to mathematics of load-sharing systems and their reliability measures. Moreover, the problem of constructing the load function in accordance with the required reliability measures is formally stated in this section. Explicit and simple expressions for synthesis the load function for the case of the Weibull probability distribution of time to failure are derived in Section 3. Cases when we can obtain a unique load function and a set of functions to ensure the required reliability measures are studied in Sections 4 and 5, respectively. An interesting case when we have to ensure a required mean time to failure by the Weibull probability distribution is considered in Section 6. Reliability analysis of parallel systems consisting of two units is provided in Section 7.

Acronym

CFRF	cumulative failure rate function
FRF	failure rate function
LF	load function
SF	survivor function

Notation

$L(t)$	LF
k	load changing after failures
$P(t)$, $P_c(t)$	SF of a system without the load and under the additional load
$\Lambda(t)$, $\mathrm{M}(t)$	CFRF of a system without the load and under the additional load
$\lambda(t)$, $\mu(t)$	FRF of a system without the load and under the additional load
$P_k(t)$	SF under condition that the load has been changed and became to be k
$x(t)$	shift function
$\bar{\lambda}(t)$, $\bar{\mu}(t)$	reduced FRF of a system without the load and under the additional load
α, a and β, b	shape and scale parameters of the Weibull distribution
T	parameter of the uniform distribution
m, m_c	mean time to failure of a system without the load and under the additional load
r, s	constant parameters used in Examples 1 and 3
Γ	gamma function

2. Formal problem statement

Let $L(t)$ be a differentiable function of time characterizing the load rate. We assume for simplicity that the condition $L(0) = 1$ has to be valid at time $t = 0$. This is an analogue of the normalizing condition which means that the system starts working under a "normal" load condition.

We also introduce another load denoted k which is changed after a unit failure at a random time in a system having several units. The LF $L(t)$ and k coincide for a system consisting of one unit, i.e., $L(t) = k$. However, if a system has a number of units, then the function $L(t)$ transforms the system SF $P(t)$ (under condition that the load k is not changed and is equal to 1 after the unit failures) into the system SF $P_c(t)$ taking into account changes of k.

The SF can be represented in the form $P(t) = e^{-\Lambda(t)}$, where $\Lambda(t) = \int_0^t \lambda(\tau)\,d\tau$ is the corresponding cumulative failure rate function (CFRF), $\lambda(t)$ is the failure rate function (FRF).

One of the most important questions of load-sharing models is a rule that governs how failure rates are changed after some units in the system fail or after changing the internal and external loads. This rule depends on the reliability application and on the system reliability behavior. We assume below that the change of the load in k times leads to change of the system failure rate in k times. It is known from the work (23) that the SF of a system under the shared load is represented also as $P_c(t) = e^{-\mathrm{M}(t)}$. Here the CFRF $\mathrm{M}(t)$ is defined as $\mathrm{M}(t) = L(t)\,\Lambda(t - x(t))$. The function $L(t)$ is a solution of the following differential equation

$$x'(t) = \frac{L'(t)}{L(t)} \frac{\Lambda(t - x(t))}{\lambda(t - x(t))}$$

under the initial condition $x(0) = 0$. The function $x(t)$ will be called the *shift function*.

In order to explain the shift function, we consider, for example, a case when the load on a system changes only once at time t_1 such that the load rate changes from the initial value 1 to k. Let $P_k(t)$ be the SF of the system under condition that the load has been changed and became to be k. The main idea underlying the computation of the SF $P_c(t)$ after the load changes is the so-called *condition of the residual lifetime conservation* of the system. Namely, it can be written as

$$P_c(t) = \begin{cases} P(t), & \text{if } t < t_1, \\ P_k(t - x), & \text{if } t \geq t_1, \end{cases}$$

where the value of the shift x is chosen such that the area under the probability density function $f(t) = -P'(t)$ in interval $[t_1, +\infty)$ would be equal to the area under the probability density function $f_k(t) = -P_k'(t)$ in interval $[t_1 - x, +\infty)$.

This is equivalent to the condition of continuity of the SF.

Rather simple expressions for $P_k(t)$ can be obtained if the load factor k increases in proportion to the system failure rate. In particular, if the load increases from 1 to k_0 simultaneously with the system failure rate increase in k_0 times, then there holds

$$P_{k_0}(t) = P^{k_0}(t).$$

Indeed, the above can be derived from the following:

$$P_{k_0}(t) = e^{-\int_0^t k_0 \lambda(x)dx} = e^{-k_0 \Lambda(t)} = P^{k_0}(t).$$

A case when changes of the load occur step-wise at discrete time instances $t_1, t_2, ..., t_m$, and the load between these instances is constant, has been studied by Gurov and Utkin (22). The loads at these instances become values $k_1, k_2, ..., k_m$, respectively. We assume $t_0 = 0$, $t_{m+1} = +\infty$, $k_0 = 1$. Then the reliability of the system, taking into account the changeable load of the above form, is

$$P_c(t) = P_{k_i}(t - x_i),$$

if $t_i \leq t < t_{i+1}$, $i = 0, 1, 2, ..., m$.

The function $P_{k_i}(t)$ is the SF of the system under condition that the corresponding load is k_i. Parameters x_i, $i = 1, 2, ..., m$, are computed by means of the recurrent algorithm from the following equation:

$$P_{k_i}(t_i - x_i) = P_{k_{i-1}}(t_i - x_{i-1}).$$

At that, $x_0 = 0$.

If a system is under the continuously varying load with the LF $L(t)$, then it has been proved (23) that the failure rate of the system taking into account the load is $\mu(t) = L(t)\lambda(t - x(t))$. It is interesting to note that this expression is similar to the expression for the function $\mathrm{M}(t)$. Therefore, for deriving the LF $L(t)$, we can apply the following method. First, we find the shift function $x(t)$ from the equality

$$\frac{\mu(t)}{\mathrm{M}(t)} = \frac{\lambda(t - x(t))}{\Lambda(t - x(t))}. \tag{1}$$

Then the LF $L(t)$ is determined as follows:

$$L(t) = \frac{\mathrm{M}(t)}{\Lambda(t - x(t))}. \tag{2}$$

Let us introduce the functions

$$\bar{\lambda}(t) = \frac{\lambda(t)}{\Lambda(t)}, \quad \bar{\mu}(t) = \frac{\mu(t)}{\mathrm{M}(t)}.$$

These functions will be called as reduced failure rates. Then equality (1) can be represented as

$$\bar{\mu}(t) = \bar{\lambda}(t - x(t)).$$

Hence

$$t - x(t) = \bar{\lambda}^{-1}(\bar{\mu}(t)).$$

According to (2), we get an expression for the LF

$$L(t) = \frac{\mathrm{M}(t)}{\Lambda\left(\bar{\lambda}^{-1}(\bar{\mu}(t))\right)}. \tag{3}$$

The above derivation has a sense under condition of the monotonicity of function $\bar{\lambda}(t)$ and under condition of existence of the inverse function. It follows from $\bar{\lambda}(t) = (\ln \Lambda(t))'$ that the function $\bar{\lambda}(t)$ is monotone if and only if the function $\ln \Lambda(t)$ is convex (or concave). This condition will be called the *load uniqueness criterion*.

So, expression (3) allows us to compute the shared LF such that the SF $P(t)$ with a convex or concave cumulative failure rate $\ln \Lambda(t)$ is transformed into a predefined SF $P_c(t)$ under this load. Another equivalent expression for computing the LF follows from (3)

$$L(t) = \frac{\mu(t)}{\lambda\left(\bar{\lambda}^{-1}(\bar{\mu}(t))\right)}. \tag{4}$$

The LF in (4) is explicitly expressed through FRFs $\lambda(t)$ and $\mu(t)$, but not through CFRFs $\Lambda(t)$ and $\mathrm{M}(t)$. At that, the implicit function $\bar{\lambda}^{-1}(\bar{\mu}(t))$ of the CFRF is obviously preserved.

3. The load function by the Weibull probability distribution

Assume that the system time to failure is governed by the Weibull probability distribution with shape and scale parameters α and β, respectively. Then we can write $P(t) = e^{-(t/\beta)^{\alpha}}$. The CFRF and FRF are

$$\Lambda(t) = (t/\beta)^{\alpha}, \quad \lambda(t) = \frac{\alpha t^{\alpha-1}}{\beta^{\alpha}}.$$

The reduced failure rate $\bar{\lambda}(t) = \alpha/t$ is monotone, and its inverse function is $\bar{\lambda}^{-1}(y) = \alpha/y$.

Suppose that the SF after load $L(t)$ becomes to be $P_c(t) = e^{-\mathrm{M}(t)}$. Then it follows from (3) that there holds

$$L(t) = \frac{\mathrm{M}(t)}{\Lambda\left(\bar{\lambda}^{-1}(\bar{\mu}(t))\right)} = \frac{\mathrm{M}(t)}{\Lambda\left(\frac{\alpha}{\bar{\mu}(t)}\right)} = \frac{\mathrm{M}(t)}{\left(\frac{\alpha}{\beta\bar{\mu}(t)}\right)^{\alpha}}.$$

Hence

$$L(t) = \left(\frac{\beta}{\alpha}\right)^{\alpha} \mathrm{M}^{1-\alpha}(t)\,\mu(t)^{\alpha}. \tag{5}$$

By using (5), we can compute the load which transforms the system with the Weibull lifetime distribution to a system with an arbitrary lifetime distribution. The shift function is obviously defined as

$$x(t) = t - \frac{\alpha}{\bar{\mu}(t)} = t - \frac{\alpha \mathrm{M}(t)}{\mu(t)}.$$

Let us prove that expression (5) is correct. In order to do that we show that the CFRF of the system under the load in the form of (5) coincides with $\mathrm{M}(t)$. It follows from the equality

$$L^{\frac{1}{\alpha}}(t) = \frac{\beta}{\alpha}\mathrm{M}^{\frac{1}{\alpha}-1}(t)\,\mu(t),$$

that

$$\int_0^t L^{\frac{1}{\alpha}}(\tau)\,d\tau = \frac{\beta}{\alpha}\int_0^t \mathrm{M}^{\frac{1}{\alpha}-1}(\tau)\,\mu(\tau)\,d\tau = \beta \mathrm{M}^{\frac{1}{\alpha}}(t).$$

It has been shown by Gurov and Utkin (23) that the CFRF of the system under the load is

$$\Lambda\left(\int_0^t L^{\frac{1}{\alpha}}(\tau)\,d\tau\right) = \left(\frac{1}{\beta}\int_0^t L^{\frac{1}{\alpha}}(\tau)\,d\tau\right)^{\alpha} = \mathrm{M}(t),$$

as was to be proved.

In particular, if the system time to failure has the exponential probability distribution with the parameter λ, then $L(t) = \mu(t)/\lambda$ and $x(t) = t - \mathrm{M}(t)/\lambda(t)$.

Let us consider another special case of the LF in (5) when the system time to failure has the Weibull distribution with parameters a and b. We can write in this case

$$P_c(t) = e^{-\left(\frac{t}{b}\right)^a}, \ \mathrm{M}(t) = \left(\frac{t}{b}\right)^a, \ \mu(t) = \frac{at^{a-1}}{b^a}.$$

By using (5), we get

$$L(t) = \left(\frac{\beta}{\alpha}\right)^{\alpha} \left(\frac{t}{b}\right)^{a(1-\alpha)} \left(\frac{at^{a-1}}{b^a}\right)^{\alpha}.$$

Hence

$$L(t) = \frac{a^{\alpha}}{b^a} \left(\frac{\beta}{\alpha}\right)^{\alpha} t^{a-\alpha}. \tag{6}$$

The shift function in this case is the following linear function $x(t) = (1 - \alpha/a)\,t$. If, for instance, the Weibull probability distribution is transformed into the Weibull distribution with the same shape parameter $a = \alpha$, but the scale parameter is reduced from β to b, then the LF must be constant $L(t) = (\beta/b)^{\alpha} > 1$. This fact can be used, for example, for reducing the testing time for estimation of the mean time to failure.

4. Unique load

Suppose that the time to failure of a system under normal working conditions (without the additional load) is governed by the Weibull distribution with the shape and scale parameters α and β, respectively. Then the load uniqueness criterion is satisfied. Let us determine the LF in order to provide the SF $P_c(t) = e^{-\mathrm{M}(t)}$ with some predefined CFRF $\mathrm{M}(t)$.

Example 1. (transforming a system into a less reliable system). Suppose that time to failure of a system under condition of the additional load has a unimodal probability distribution with the failure rate $\mu(t) = (t-r)^2 + s$, where $r, s > 0$ are some constant parameters. Then we can write

$$\mathrm{M}(t) = \frac{1}{3}(t-r)^3 + st + \frac{1}{3}r^3.$$

According to (5), we get the LF in the explicit form:

$$L(t) = \left(\frac{\beta}{\alpha}\right)^{\alpha} \left(\frac{1}{3}(t-r)^3 + st + \frac{1}{3}r^3\right)^{1-\alpha} \left((t-r)^2 + s\right)^{\alpha}.$$

The functions $L(t)$ for parameters $\beta = 10$, $r = 2$, $s = 1$ by three values $\alpha = 2.5$, $\alpha = 3$ and $\alpha = 3.5$ are shown in Fig. 1.

In sum, in order to get a system having time to failure with a certain probability distribution and with the unimodal FRF, the load on the system has to behave in accordance with Fig. 1 depending on the parameter α. Starting from some time moment, the load has to be constant (for $\alpha = 3$), or increasing (for $\alpha < 3$), or decreasing (for $\alpha > 3$).

Fig. 2 illustrates the SF for $\alpha = 3$ without the load ($L(t) = 1$) and under the additional load $L(t)$. As a result of the load the system becomes almost unreliable. The SF tends to zero during the starting period of the system working.

Example 2. (transforming a system into a more reliable system). Suppose that the time to failure of a system under condition of additional load has the uniform probability distribution in interval from 0 to $T = 100$. Then we can write $P_c(t) = 1 - t/T$ for $t < T$. The CFRF is $\mathrm{M}(t) = -\ln(1 - t/T)$, the failure rate is $\mu(t) = 1/(T - t)$.

According to (5), we get the LF which transform the initial SF $P(t)$ into the function $P_c(t)$:

$$L(t) = \left(\frac{\beta}{\alpha}\right)^{\alpha} \left(-\ln\left(1 - \frac{t}{T}\right)\right)^{1-\alpha} \left(\frac{1}{T-t}\right)^{\alpha}.$$

Curves of the LF are shown in Fig. 3 by the parameters: $\alpha = 3$, $\beta = 10$, $T = 100$.

One can see from Fig. 3 the required SF can be obtained if the LF will quickly decrease, and it will be very close to zero by $t \geq 5$. It can also be seen that the behavior of the LF significantly differs from the same behavior shown in Fig. 1.

SFs $P(t)$ and $P_c(t)$ are shown in Fig. 4. One can see from the figure that the system becomes more reliable.

5. Non-unique load

Let us study a case when the SF can be transformed by means of different LFs, i.e., we have a non-unique LF. According to the load uniqueness criterion, this case corresponds to a non-monotone reduced failure rate $\bar{\lambda}(t)$.

Example 3. Let $\lambda(t) = (t-r)^2 + s$. Then there holds $\Lambda(t) = \frac{1}{3}(t-r)^3 + st + \frac{1}{3}r^3$. We find the reduced failure rate

$$\bar{\lambda}(t) = \frac{\lambda(t)}{\Lambda(t)} = \frac{3(t-r)^2 + 3s}{(t-r)^3 + 3st + r^3},$$

or

$$\bar{\lambda}(t) = \frac{3(t-r)^2 + 3s}{t(t^2 - 3rt + 3r^2 + 3s)}.$$

This function by $r = 2$, $s = 1$ is depicted in Fig. 5.

The LF $L(t)$ is ambiguously determined. In order to obtain one of the possible LFs, we have to allocate a "unique" part denoted as $\bar{\lambda}_0(t)$ from the function $\bar{\lambda}(t)$ and to determine the corresponding LF $L_0(t)$ by means of (3). It is obvious that there exist infinitely many functions $L_0(t)$, and the final LF is ambiguously determined.

As an example, we can take the function $\bar{\lambda}_0(t)$ such that its inverse function $t = \bar{\lambda}_0^{-1}(y)$ satisfies the following properties:

1. if $y \geq 0.4$, then t is a root of equation $y = \bar{\lambda}(t)$ in interval $[0, 2]$;
2. if $y < 0.4$, then t is a root of equation $y = \bar{\lambda}(t)$ in interval $[8, +\infty]$.

In this case, the LF $L_0(t)$ has a discontinuity at point t_0 such that $\bar{\mu}(t_0) = 0.4$. The corresponding shift function $x_0(t)$ is also discontinuous at the same point.

6. Mean time to failure by the Weibull probability distribution

If the time to failure of a system without the load and under the additional load is governed by the Weibull probability distribution with parameters (a, β) and (a, b), respectively, then it follows from (6) that the load has the form of the power function

$$L(t) = \left(\frac{a}{\alpha}\right)^{\alpha} \frac{(t/b)^a}{(t/\beta)^{\alpha}}.$$

Since the mean time to failure of the system without the load and under the additional load are $m = \beta\Gamma(1 + 1/a)$ and $m_c = b\Gamma(1 + 1/a)$, respectively, then there holds

$$L(t) = \left(\frac{a}{\alpha}\right)^{\alpha} \frac{((t/m_c)\Gamma(1 + 1/a))^a}{((t/m)\Gamma(1 + 1/\alpha))^{\alpha}}.$$

For the identical shape parameters $\alpha = a$, we can see that the LF is constant. It is expressed through the expectation of time to failure, i.e., there holds

$$L(t) = (m/m_c)^{\alpha} = \text{const.}$$

So, the requirement to increase (to reduce) the system mean time to failure in d times implies that the load has to be reduced (increased) in d^{α} times. If the shape parameters of the Weibull probability distributions are different, i.e., $\alpha \neq a$, then the corresponding load is not constant, but a function of time.

7. A parallel load-sharing system

Let us analyze a two-unit parallel system consisting of identical units. The following system states can be defined:

State 1: Two units are working.

State 2: Only one unit is working.

State 3: Both units have failed, i.e., system is in the failure state.

It is shown by Xie et al. (24) that if the failure rate of a single unit is λ, then the SF of the parallel system is $P(t) = 2e^{-\lambda t} - e^{-2\lambda t}$. We suppose that the system is load-sharing, i.e., the load on the working units increases and is equal to k after failure of a unit. If $k > 1$, then the load is increased. If $k = 1$, then the load is not changed. As a result of the load changes, the SF $P_c(t)$ is changed. The SFs $P(t)$ and $P_c(t)$ are supposed to be known. Our aim is to determine the load function $L(t)$ of the parallel system such that its impact on the system is equivalent to the impact of the unit failures with the load k. In other words, we say about the possible replacement of the random load on units after failures of other units by non-random LF $L(t)$.

It should be noted that in contrast to most works devoted to m-out-of-n systems, we suppose that the load k can be arbitrary.

The CFRF and the FRF of the parallel system are

$$\Lambda(t) = -\ln\left(2e^{-\lambda t} - e^{-2\lambda t}\right), \quad \lambda(t) = \frac{2\lambda\left(e^{-\lambda t} - e^{-2\lambda t}\right)}{2e^{-\lambda t} - e^{-2\lambda t}},$$

respectively. Then the reduced failure rate is determined as

$$\bar{\lambda}(t) = \frac{-2\lambda\left(e^{-\lambda t} - e^{-2\lambda t}\right)}{\left(2e^{-\lambda t} - e^{-2\lambda t}\right)\ln\left(2e^{-\lambda t} - e^{-2\lambda t}\right)}. \tag{7}$$

It is easy to show that this function is a monotone decreasing function. However, the inverse function can be only numerically found.

Let us determine the SF of the system under condition of the load which is caused by failures

$$P_c(t) = e^{-2\lambda t} + 2\int_0^t f(t_1)\,\bar{F}(t,t_1)\,dt_1,$$

where $f(t)$ is the probability density function of time to unit failure; $\bar{F}(t,t_1)$ is the SF of a unit under condition that the SF of time to the second unit failure has been changed at time t_1.

Similarly to the same derivation given in work by Gurov and Utkin (22), we get

$$\bar{F}(t,t_1) = e^{-\lambda(t_1 + k(t-t_1))} \ \text{ for } t \geq t_1.$$

Then

$$P_c(t) = e^{-2\lambda t} + 2\lambda \int_0^t e^{-\lambda t_1} e^{-\lambda(t_1 + k(t-t_1))} dt_1$$
$$= e^{-2\lambda t} + 2\lambda e^{-k\lambda t} \int_0^t e^{-\lambda(2-k)t_1} dt_1.$$

Consequently, there holds

$$P_c(t) = \begin{cases} (1+2\lambda t)\, e^{-2\lambda t}, & k = 2, \\ \frac{ke^{-2\lambda t} - 2e^{-k\lambda t}}{k-2}, & k \neq 2. \end{cases}$$

Let us compute the LF transforming the SF $P(t)$ into $P_c(t)$. In order to carry out that, we compute all functions from (3), which depend on the load. If $k = 2$, then the CFRF and the FRF of the parallel system under the load are

$$\mathrm{M}(t) = -\ln\left((1+2\lambda t)\, e^{-2\lambda t}\right), \;\; \mu(t) = \frac{4\lambda^2 t}{1+2\lambda t},$$

respectively. The reduced failure rate is of the form:

$$\bar{\mu}(t) = \frac{-4\lambda^2 t}{(1+2\lambda t)\ln\left((1+2\lambda t)\, e^{-2\lambda t}\right)}.$$

According to (3), we get the system LF

$$L(t) = \frac{\mathrm{M}(t)}{\Lambda\left(\bar{\lambda}^{-1}(\bar{\mu}(t))\right)} = \frac{\ln\left((1+2\lambda t)\, e^{-2\lambda t}\right)}{\ln\left(2e^{-\lambda u} - e^{-2\lambda tu}\right)}. \tag{8}$$

Here $u = \bar{\lambda}^{-1}(\bar{\mu}(t))$.

If $k \neq 2$, then the CFRF and the FRF of the parallel system under the load are

$$\mathrm{M}(t) = -\ln\left(\frac{2e^{-k\lambda t} - ke^{-2\lambda t}}{2-k}\right), \; \mu(t) = \frac{2k\lambda\left(e^{-k\lambda t} - e^{-2\lambda t}\right)}{2e^{-k\lambda t} - ke^{-2\lambda t}},$$

respectively. The reduced failure rate is of the form:

$$\bar{\mu}(t) = \frac{-2k\lambda\left(e^{-k\lambda t} - e^{-2\lambda t}\right)}{\left(2e^{-k\lambda t} - ke^{-2\lambda t}\right)\ln\left(\frac{2e^{-k\lambda t} - ke^{-2\lambda t}}{2-k}\right)}.$$

The LF is determined from (3) as follows:

$$L(t) = \frac{\mathrm{M}(t)}{\Lambda\left(\bar{\lambda}^{-1}(\bar{\mu}(t))\right)} = \frac{\ln\left(\frac{2e^{-k\lambda t} - ke^{-2\lambda t}}{2-k}\right)}{\ln\left(2e^{-\lambda u} - e^{-2\lambda u}\right)}. \tag{9}$$

Here $u = \bar{\lambda}^{-1}(\bar{\mu}(t))$. The function $\bar{\lambda}(t)$ is obtained from (7).

In order to use (8) and (9), we have to find limits of the above functions by $t \to 0$ and $t \to \infty$. The corresponding expressions are given in Table 1.

It follows from Table 1 that the limits of the function which is inverse for $y = \bar{\lambda}(t)$ are

$$\bar{\lambda}^{-1}(y) \sim \frac{1}{y}, \;\text{ by }\; y \to 0, \tag{10}$$

Table 1. Limits of some functions for the standby system

Functions	$t \to 0$	$t \to +\infty$
$\Lambda(t)$	$\lambda^2 t^2$	λt
$\lambda(t)$	$2\lambda^2 t$	λ
$\bar{\lambda}(t)$	$2/t$	$1/t$
$\mathrm{M}(t)$	$k\lambda^2 t^2$	$\lambda t \cdot \min(k,2)$
$\mu(t)$	$2k\lambda^2 t$	$\lambda \cdot \min(k,2)$
$\bar{\mu}(t)$	$2/t$	$1/t$
$u = \bar{\lambda}^{-1}(\bar{\mu}(t))$	$\bar{\lambda}^{-1}(2/t) \sim t$	$\bar{\lambda}^{-1}(1/t) \sim t$
$L(t)$	k	$\min(k,2)$

$$\bar{\lambda}^{-1}(y) \sim \frac{2}{y}, \text{ by } y \to +\infty. \tag{11}$$

Therefore, if $t \to 0$, then, according to (10), we have

$$u = \bar{\lambda}^{-1}(\bar{\mu}(t)) = \bar{\lambda}^{-1}\left(\frac{2}{t}\right) \sim t.$$

If $t \to +\infty$, then, according to (9), there holds

$$u = \bar{\lambda}^{-1}(\bar{\mu}(t)) = \bar{\lambda}^{-1}\left(\frac{1}{t}\right) \sim t.$$

LFs for the case $\lambda = 0.1$ and for values $k = 1.5$, $k = 2$, $k = 3$ are depicted in Fig. 6. One can explicitly see some regularity of the behavior of $L(t)$. The load by $k \leq 2$ is non-monotonically changed. It decreases starting from the value k. Then it increases and again tends to the value k. We see that the smallest values of the LFs by $k \leq 2$ correspond to the time moments which are close to the mean time to failure of a unit $1/\lambda = 10$. In other words, the LF has to be decreased before the first unit failure in order to satisfy the condition of the residual lifetime conservation of the system by small values of k, the function $L(t)$. This time interval can be regarded as a transient period. By $k > 2$, the LF monotonically decreases starting from the value k till 2. Limits of the function $L(t)$ correspond to Fig. 6.

SFs of the parallel system for values $k = 1$, $k = 1,5$, $k = 2$, $k = 3$ are shown in Fig. 7. It is obvious that the increase of the load on the working unit after failure of the first unit implies reduction of the system reliability. The case $k = 1$ corresponds to the system working under the initial condition without the additional shared load. At that, the function $L(t)$ is equal to 1.

So, if the load on a working unit is random in the sense that it is 1 before the first failure, and it is k after this failure, then the system LF $L(t)$ has a form shown in Fig. 6. This implies that the resulting SF is the same by the random load and by the determined load.

8. Conclusion

The reliability analysis of load-sharing systems under different loads has been provided in the paper. In contrast to many papers devoted to this topic, we have studied the inverse problem of determining the load function by having requirements concerning the system SF. In fact, the solved problem can be regarded as a system synthesis problem because we synthesize the load function in accordance with the required system SF.

It should be noted that the problem of the load function construction in a general case can be solved only numerically. However, it has been shown that explicit or very simple expressions can be obtained in many special cases. For example, when the system times to failure without load and with the additional load are governed by the Weibull probability distribution with predefined shape and scale parameters, respectively, we get quite simple explicit expressions for $L(t)$. Simple expressions can be also obtained for the parallel systems when the unit time to failure is exponentially distributed.

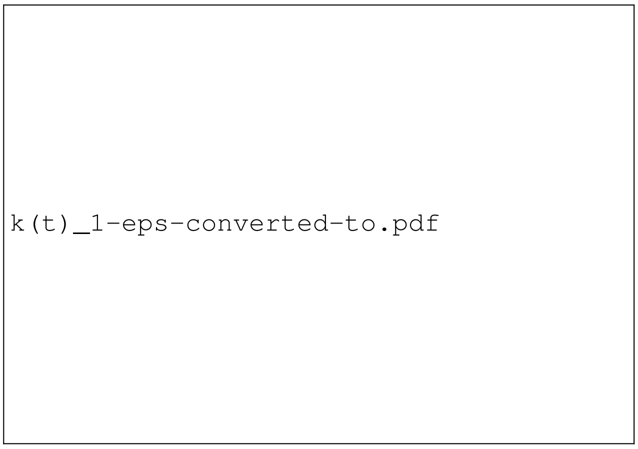

Fig. 1. Functions $L(t)$ by three values of α

Fig. 2. SFs $P\left(t\right)$ and $P_c\left(t\right)$

Fig. 3. The function $L(t)$ leading to a more reliable system

Fig. 4. SFs $P\,(t)$ and $P_c\,(t)$

Fig. 5. The reduced failure rate by the non-unique load

Fig. 6. Functions $L(t)$ by $k = 1.5$, $k = 2$, $k = 3$

k(t)_8-eps-converted-to.pdf

Fig. 7. SFs of the standby system for values $k = 1$, $k = 1,5$, $k = 2$, $k = 3$

Of course, only simple cases of the system reliability have been studied in the paper. Other systems and probability distributions could be investigated. This is a direction for further research.

The various numerical examples have illustrated the proposed methods for constructing the LF and for analyzing the system reliability under different conditions of working.

Another important question which has to be stated is how to implement the LF $L(t)$ which provides a required SF $P_c(t)$. This is a very difficult problem whose solution depends on a special application. If, for example, we return to electrical network (1; 2; 3) mentioned above, then the electricity load can be controlled in accordance with the obtained function $L(t)$. In queueing systems with buffers of a certain capacity for task allocation studied by Huang and Xu (21), the required load function can be realized by selection of the capacity of buffers. However, the paper does not give an answer how to realize a special system with the obtained LF. This is also a direction for further research.

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